Spectral analysis of terrestrial planets and gas giants

Gas giants have a discrete (acoustic) spectrum, part III

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Recall from the first lecture

The acousto-gravitational system of equations (equation (11) in the first lecture)

$$\partial_t^2(\rho_0 u) + 2\Omega \times \partial_t(\rho_0 u) = \nabla[c^2\,(\nabla\cdot(\rho_0 u) - \tilde{s}\cdot u)] - (\nabla\cdot(\rho_0 u))g_0' - \rho_0 \nabla S(\rho_0 u). \tag{1}$$

The equation comes together with the free-surface boundary condition

$$[\nabla \cdot u]|_{\partial M} = 0.$$

So far we have assumed the planet to be artificially truncated so that $c^2 > 0$.

In this lecture we let the sound speed c go zero at ∂M. We will see that to the leading order the equation becomes a wave equation for a Riemannian metric which is singular at ∂M.

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Extracting the Laplacian

- neglecting the Corriolis term and the lower-order terms the acousto-gravitational system becomes
$$\partial_t^2(\rho_0 u) = \nabla[c^2(\nabla \cdot (\rho_0 u) - \tilde{s} \cdot u)] = \nabla[c^2 T u]$$
- we use the generalized Helmholtz decomposition (Lemma 2 in the first lecture) to write $u = T^*(\rho_0\varphi) + u_2$ with $\varphi \in H^1(M)$ and $u_2 \in \mathrm{Ker}(T)$; the equation becomes
$$\partial_t^2(T^*(\rho_0\varphi)) + \partial_t^2 u_2 = \rho_0^{-1}\nabla[c^2 T T^*(\rho_0\varphi)].$$
- setting $u_2 = 0$ and replacing ∇ with T^* to obtain
$$\partial_t^2(T^*(\rho_0\varphi)) = \rho_0^{-1} T^*[c^2 T T^*(\rho_0\varphi)]$$
- For a moment assume ρ_0 to be constant. Then ignoring the lower order terms, we pull T^* out of the equation to get
$$\partial_t^2(\rho_0\varphi) = c^2\rho_0^{-1} T T^*(\rho_0\varphi) = \underbrace{c^2\Delta}_{\Delta_c}(\rho_0\varphi) + \text{ l.o.t.}$$

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Riemannian setting

The Laplacian Δ_c is really the Laplace-Beltrami operator of the Riemannian metric $c^{-2}\mathrm{Id}_{n\times n}$.

Consider a bounded smooth domain $\Omega \subseteq \mathbb{R}^n$. We use coordinates (x, y) in a neighborhood of the boundary where $\partial\Omega = \{x = 0\}$ and $y = (y_1, \ldots, y_{n-1})$ are coordinates on $\partial\Omega$.

Generalized setting: Replace the conformal sound speed by an anisotropic one, represented as a Riemannian metric g. The vanishing of the sounds speed forces the metric to be of the form

$$g = \frac{\bar{g}}{x^\alpha} = \frac{dx^2 + h_{ij}(x, y)dy^i dy^j}{x^\alpha}$$

where $\alpha \in (0, 2)$ and $\bar{g}$ is a Riemannian metric that is smooth up to and including $\partial\Omega$ and $h(x, \cdot)$ is a family of Riemannian metrics on $\partial\Omega$.

The geometry of g is called **gas giant geometry**.

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Degenerate Laplacian

The Laplace-Beltrami operator of the gas giant metric $g(x, y)$ is

$$\Delta_g u = \sum_{i,j} \frac{1}{\sqrt{\det(g)}} \partial_i \left(\sqrt{\det(g)} g_{ij}^{-1} \partial_j u \right).$$

To derive a more manageable formula for Δ_g we use the fact that $\det(g) = x^{-\alpha n} \det(\bar{g})$. Then, using the fact that $\bar{g}_{00} = 1$ and $\bar{g}_{0i} = 0$ for $i \neq 0$, we obtain

$$\begin{aligned}\Delta_g u &= \sum_{i,j} \frac{x^{\alpha \frac{n}{2}}}{\sqrt{\det(\bar{g})}} \partial_i \left(x^{\alpha(1-\frac{n}{2})} \sqrt{\det(\bar{g})} \bar{g}_{ij}^{-1} \partial_j u \right) \\ &= \frac{x^{\alpha \frac{n}{2}}}{\sqrt{\det(\bar{g})}} \left[\partial_x \left(x^{\alpha(1-\frac{n}{2})} \sqrt{\det(\bar{g})} \partial_x u \right) + \sum_{i,j} \partial_{y_i} \left(x^{\alpha(1-\frac{n}{2})} \sqrt{\det(\bar{g})} \bar{g}_{ij}^{-1} \partial_{y_j} \right) \right] \\ &= x^{\alpha} \Delta_{\bar{g}} u + \alpha \left(1 - \frac{n}{2} \right) x^{\alpha - 1} \partial_x u.\end{aligned}$$

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Uniformly degenerate partial differential operators

The Laplace-Beltrami operator is degenerate at $\{x = 0\}$, but the degeneracy is uniform in the basic vector fields $x\partial_x$ and $x\partial_{y_k}$. In particular, after multiplying with the power $x^{2-\alpha}$ we get

$$x^{2-\alpha}\Delta_g = (x\partial_x)^2 u + \det(\bar{g})^{-1/2}[(x\partial_x)\det(\bar{g})^{1/2}][(x\partial_x)u] \\ + \det(\bar{g})^{-1/2}\sum_{i,j}(x\partial_{y_i})\big(\det(\bar{g})^{1/2}\bar{g}_{ij}^{-1}(x\partial_{y_j})u\big).$$

In general, a 0-**differential operator** P of order m is of the form

$$P = \sum_{i+|\beta|\leq m} a_{i,\beta}(x,y)(x\partial_x)^i(x\partial_{y_\beta})^\beta \tag{2}$$

where $a_{i,\beta}(x,y)$ are smooth functions. Clearly $x^{2-\alpha}\Delta_g$ is a 0-differential operator of order 2.

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A brief introduction to 0-calculus

A more detailed exposition on the basics of 0-calculus can be found in [Maz91].

The **indicial operator** of the 0-differential operator P in (2) is

$$I(P) = \sum_{i \leq m} a_{i,0}(0, y)(x\partial_x)^i.$$

In particular, for the Laplacian $L = x^{2-\alpha}\Delta_g$,

$$I(L) = (x\partial_x)^2 - \alpha\left(\frac{n}{2} - 1\right)(x\partial_x).$$

We will use the indicial operator $I(L)$ momentarily to compute an invariant called indicial roots (the indicial operator fails to be invertible).

Associated to a 0-differential operator P is its 0-**principal symbol** ${}^0\sigma_m(P)$, which is obtained by replacing $x\partial_x$ and $x\partial_{y_k}$ with ξ and η_k in the local presentation of the operator.

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An operator is 0-**elliptic**, if its principal symbol is non-vanishing when $(\xi, \eta) \neq (0,0)$.

Recalling that $\bar{g}_{ij} = h_{ij}$ when $i,j \neq 0$, the 0-principal symbol of the Laplacian L is

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Hence L is 0-elliptic. As such, the calculus of 0-pseudodifferential operators offers analogues of all the familiar constructions in pseudodifferential theory.

There is an elliptic parametrix construction for L, and the various properties of the parametrix G for L obtained through this construction lead to sharp mapping and regularity properties.

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- in slightly more detail, parametrices of 0-differential operators are 0-pseudodifferential operators, which are usually defined on the level of their Schwartz kernels
- the Schwartz kernel of a 0-pseudodifferential operator has conormal singularities on the diagonal and are smooth elsewhere, the same as classical pseudodifferential operators
- The difference is what happens at the boundary of the domain, which classical theory does not account for
- roughly speaking, the Schwartz kernel of a 0-pseudodifferential operator has polyhomogeneous conormal behavior at the boundary hypersurfaces $\partial\Omega \times \Omega$, $\Omega \times \partial\Omega$ and the front face
- the part here that is rough is that the natural space for the Schwartz kernel is a 0-double tangent space obtained blowing up the boundary diagonal $\mathrm{diag}(\partial\Omega \times \partial\Omega)$ in $\Omega \times \Omega$
- polyhomogeneity means that there are expansions into asymptotic sums of terms $x^z(\log x)^k a_{z,k}(y)$ when $\{x = 0\}$ defines a boundary hypersurface

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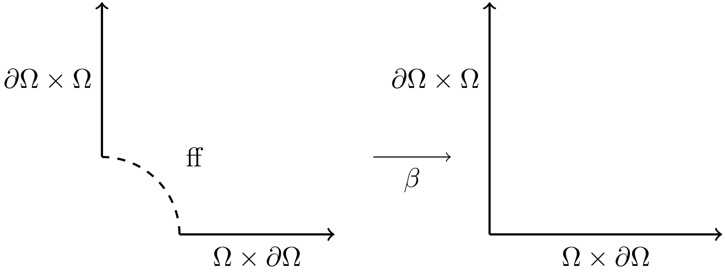

Figure 1: An example of a blow-up: Here $\Omega = [0, \infty)$ and on the left we have blown up the boundary diagonal in $\Omega \times \Omega$, i.e., the origin. This corresponds to introduction of polar coordinates β around $(0, 0)$. Pseudodifferential operator in the 0-calculus have polyhomogeneous asymptotics at $\Omega \times \partial\Omega$, $\partial\Omega \times \Omega$ and the front face ff.

Self-adjointness

Consider the densely defined unbounded operator

$$\Delta_g \colon L^2(\Omega, dV_g) \longrightarrow L^2(\Omega, dV_g). \tag{3}$$

This is symmetric on the core domain $C_0^\infty(\Omega)$ of smooth functions compactly supported in the interior.

Before studying the spectrum of Δ_g we need to determine whether Δ_g has a unique self-adjoint extension, or if boundary conditions need to be imposed to obtain a self-adjoint realization.

This is where we need to compute the indicial roots of Δ_g. The pair of indicial roots, $\gamma_\pm$, are the values of γ so that $I(x^{2-\alpha}\Delta_g)x^\gamma = 0$ (the indicial operator fails to be invertible). These are the exponents which yield approximate solutions in the sense that

$$\Delta_g x^\gamma = \mathcal{O}(x^{\gamma-1+\alpha})$$

rather than the expected rate $\mathcal{O}(x^{\gamma-2+\alpha})$, i.e., there is leading order cancellation.

Self-adjointness

Consider the densely defined unbounded operator

$$\Delta_g \colon L^2(\Omega, dV_g) \longrightarrow L^2(\Omega, dV_g). \tag{3}$$

This is symmetric on the core domain $C_0^\infty(\Omega)$ of smooth functions compactly supported in the interior.

Before studying the spectrum of Δ_g we need to determine whether Δ_g has a unique self-adjoint extension, or if boundary conditions need to be imposed to obtain a self-adjoint realization.

This is where we need to compute the indicial roots of Δ_g. The pair of indicial roots, $\gamma_\pm$, are the values of γ so that $I(x^{2-\alpha}\Delta_g)x^\gamma = 0$ (the indicial operator fails to be invertible). These are the exponents which yield approximate solutions in the sense that

$$\Delta_g x^\gamma = \mathcal{O}(x^{\gamma-1+\alpha})$$

rather than the expected rate $\mathcal{O}(x^{\gamma-2+\alpha})$, i.e., there is leading order cancellation.

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Self-adjointness and indicial roots

To calculate the indicial roots, we compute

$$\Delta_g x^\gamma = (\gamma(\gamma-1) - \alpha(n/2-1)\gamma)\, x^{\gamma-2+\alpha} + \mathcal{O}(x^{\gamma-1+\alpha}),$$

and hence γ must satisfy $\gamma^2 - (\alpha(n/2-1)+1)\gamma = 0$, or

$$\begin{aligned}\gamma_\pm &= 0, \alpha(n/2-1)+1 \\ &= \frac{1}{2}(\alpha(n/2-1)+1) \pm \frac{1}{2}(\alpha(n/2-1)+1).\end{aligned}$$

This last expression is included to emphasize the symmetry of $\gamma_\pm$ around their average, which is useful below.

Self-adjointness and indicial roots

Next, observe that a function x^γ lies in $L^2(dV_g)$ near $x = 0$ if and only if

$$\gamma > \frac{1}{2}(n\alpha/2 - 1).$$

We call this threshold the "L^2 cutoff weight". Recall $dV_g = x^{-\alpha\frac{n}{2}} dx dV_h$.

It is most natural to let Δ_g act on the Sobolev spaces adapted to the 0-vector fields:

$$H_0^k(\Omega, dV_g) = \{u : V_1 \dots V_\ell u \in L^2(dV_g) \ \ \ell \le k, \ \ \text{each } V_i \in \mathcal{V}_0(\Omega)\},$$

and their weighted version $x^\mu H_0^k = \{u = x^\mu v : v \in H_0^k\}$.

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Self-adjointness and indicial roots

It is clear from this definition that

$$\Delta_g \colon x^{\mu} H_0^2 \longrightarrow x^{\mu-2+\alpha} L^2$$

is bounded for every μ. In particular, $\Delta_g u \in L^2$ if $u \in x^{\mu} H_0^2$ where $\mu \geq 2 - \alpha$.

Since $C_0^{\infty}(\Omega)$ is dense in $x^{2-\alpha} H_0^2$, it is clear that the minimal domain, i.e., the minimal closed extension from the core domain, of (3) is contained in $x^{2-\alpha} H_0^2$.

Using the parametrix for Δ_g alluded to above, it can be proved that this is an equality:

$$\mathscr{D}_{\min}(\Delta_g) = x^{2-\alpha} H_0^2(\Omega, dV_g).$$

On the other hand, we also define the maximal domain $\mathscr{D}_{\max}(\Delta_g) = \{u \in L^2 : \Delta_g u \in L^2\}$.

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Self-adjointness and indicial roots

Proposition 1 (Self-adjointness)

The operator Δ_g is essentially self-adjoint on L^2, i.e.,

$$\mathscr{D}_{\min}(\Delta_g) = \mathscr{D}_{\max}(\Delta_g)$$

if and only if $\alpha > 2/n$.

The value $\alpha = 2/n$ is precisely the critical value where the Riemannian volume of M becomes infinite: $\mathrm{Vol}_g(\Omega) < \infty$ iff $\alpha < 2/n$.

Proof of Proposition 1

- the key issue is whether either of the indicial roots $\gamma_\pm$ lie in the critical weight interval

$$\mu_- \coloneqq \frac{1}{2}(n\alpha/2 - 1) \leq \mu \leq \frac{1}{2}(n\alpha/2 - 1) + 2 - \alpha \eqqcolon \mu_+$$

- the relevance of whether the indicial roots are included in the critical weight interval is that, using the parametrix carefully, one can deduce that if $\gamma_\pm$ do not lie in this critical weight interval, then $u \in L^2$ and $\Delta_g u \in L^2$ imply that $u \in x^{2-\alpha} H_0^2 = \mathcal{D}_{\min}$
- notice that the midpoint of this critical interval is

$$\frac{1}{2}(n\alpha/2 - 1) + 1 - \frac{1}{2}\alpha = \frac{1}{2}(\alpha(n/2 - 1) + 1)$$

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- the width of this weight interval is $2-\alpha$, whereas $\gamma_+ - \gamma_- = \alpha(n/2-1)+1$. We claim that

$$\gamma_- < \mu_- < \mu_+ < \gamma_+$$

precisely when $\alpha > 2/n$, which is verified by noting that $\alpha(n/2-1)+1 > 2-\alpha$ precisely then

- however, when $\alpha \leq 2/n$, then we can only deduce that

$$u(x,y) \sim \sum a_j(y) x^{\gamma_- + j} + \sum b_j(y) x^{\gamma_+ + j} \tag{4}$$

- this asymptotic expansion has some complicating features, such as that if $a_0 \neq 0$, then the coefficients a_j, b_j may only have finite regularity (and will have negative Sobolev regularity for large j)
- conversely, there exists a solution of $\Delta_g u = 0$ where u has an expansion of this type with any prescribed smooth leading coefficient $a_0(y)$; in any case, the upshot is that the maximal domain is far bigger than the minimal domain in this case

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The spectrum is discrete

Proposition 2 (Discrete spectrum)

Let $\mathscr{D}$ be a domain of self-adjointness for Δ_g. Then $(\Delta_g, \mathscr{D})$ is a Fredholm operator on L^2 with discrete spectrum.

When $\alpha < 2/n$, there are many possible self-adjoint extensions. The one we use, is the Dirichlet extension, corresponds to the choice of domain $\mathscr{D}_{\mathrm{Dir}}$ consisting of those $u \in L^2$ such that $\Delta_g u \in L^2$ and where the leading coefficient $a_0(y)$ in expansion (4) vanishes.

Other self-adjoint extensions correspond to other types of conditions on the pair of leading coefficient $(a_0(y), b_0(y))$ in (4). Some of the other standard ones are the Neumann extension, where $b_0(y) \equiv 0$, and the family of Robin extensions, corresponding to conditions of the form $A(y)a_0(y) + B(y)b_0(y) \equiv 0$, where A, B are given smooth functions.

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Proof of proposition 2

- The first step is to show that this operator is Fredholm. This follows from the existence of its parametrix.
- The parametrix G of Δ_g is a 0-pseudodifferential operator of order -2 which maps L^2 onto $\mathscr{D}$ (possibly modulo compact errors), and which satisfies

$$G \circ \Delta_g = \mathrm{Id} - R_1, \quad \text{and} \quad \Delta_g \circ G = \mathrm{Id} - R_2,$$

  where R_1 and R_2 are compact operators on L^2 and on $\mathscr{D}$ (with its graph topology) respectively.
- As noted earlier, the construction of this parametrix is one of the standard consequences of 0-ellipticity; details are given in [Maz91]. When $\alpha < 2/n$, a slightly more intricate construction is needed which incorporates the choice of boundary conditions; this appears in [MV14].

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- The key point here is that the operator G is constructed as an element of the 0-pseudodifferential calculus:
  - The Schwartz kernel is well-understood, as a distribution on $\Omega \times \Omega$.
  - It has asymptotic expansions at the boundary faces of the product, and equally explicit expansion near the corner of Ω^2 (which is $\partial\Omega \times \partial\Omega$).
- The upshot, however, is that it then follows by general properties of such pseudodifferential operators proved in [Maz91] that G is bounded on L^2. Of course, as a pseudoinverse to Δ_g, its range must lie in $\mathscr{D}$.
- Since G is a 0-pseudodifferential operator of order -2, it is clear that the elements in $G(L^2)$ have two derivatives in L^2, at least in the interior of Ω.
- Slightly more is true: for any $f \in L^2$ and any two vector fields $V_1, V_2 \in \mathcal{V}_0(\Omega)$, we must have that $V_1 V_2(Gf) \in x^\varepsilon L^2$ for some fixed $\varepsilon > 0$ which is independent of f.

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- A few more words about this parametrix construction, particularly when $\alpha > 2/n$. Write $\Delta_g = x^{\alpha/2-1} L x^{\alpha/2-1}$; as noted earlier, L is an elliptic 0-operator. The singular factor has been distributed on opposite sides of L to preserve symmetry.
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- A few more words about this parametrix construction, particularly when $\alpha > 2/n$. Write $\Delta_g = x^{\alpha/2-1} L x^{\alpha/2-1}$; as noted earlier, L is an elliptic 0-operator. The singular factor has been distributed on opposite sides of L to preserve symmetry.
- Let $\overline{G}$ be a parametrix for L as constructed in [Maz91]. Thus

$$\mathrm{Id} - L\overline{G} = R_1' \quad \text{and} \quad \mathrm{Id} - \overline{G}L = R_2'$$

Proof of proposition 2

- The error terms R'_1, R'_2 are operators with smooth kernels on the interior of $\Omega \times \Omega$, and which admit classical expansions at all boundary faces of a certain resolution (or blow-up) of this product, with coefficients in these expansion smooth functions on the corresponding boundary faces.
- We then write $G = x^{1-\alpha/2}\overline{G}x^{1-\alpha/2}$, so that $\Delta_g \circ G = \mathrm{Id} - x^{\alpha/2-1}R_1 x^{\alpha/2-1} = I - R_1$, $G \circ \Delta_g = \mathrm{Id} - x^{\alpha/2-1}R'_2 x^{\alpha/2-1} = \mathrm{Id} - R_2$.
- These remainder terms are much better, inasmuch as they have smooth Schwartz kernels which have polyhomogeneous expansions at the two boundary hypersurfaces of Ω^2, without need for the resolution (or blow-up) process.
- If $\Delta_g u = f \in L^2$, then applying G, we get that $u = R_1 u + Gf = R_1 u + x^{1-\alpha/2}\overline{G}x^{1-\alpha/2}f$.
- The first term is polyhomogeneous on Ω, and decays at a fixed rate strictly greater than the L^2 cutoff. When $\alpha > 2/n$, the range of G lies in $x^{2-\alpha}H^2_0$.
- This range is identified with the domain of self-adjointness $\mathscr{D}$ (again, when $\alpha > 2/n$), hence, as described above, $\mathscr{D} \subset L^2$ is indeed compact.

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