Li-Yau and Harnack estimates for nonlocal diffusion problems

Rico Zacher

Institute of Applied Analysis
Ulm University, Germany

partly joint work with
S. Kräss, A. Spener and F. Weber

1. Introduction: Classical Li-Yau and CD inequality

The classical Li-Yau inequality

Suppose $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ solves $\partial_t u - \Delta u = 0$. Then

$$-\Delta(\log u) \leq \frac{d}{2t} \quad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{1}$$

Since $\partial_t(\log u) - \Delta(\log u) = |\nabla(\log u)|^2$, this is equivalent to

$$\partial_t(\log u) \geq |\nabla(\log u)|^2 - \frac{d}{2t} \quad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{2}$$

- This extends to complete d-dimensional Riemannian manifolds M with $Ric(M) \geq 0$ (Li, Yau, Acta Math. 1986).

The classical Li-Yau inequality

Suppose $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ solves $\partial_t u - \Delta u = 0$. Then

$$-\Delta(\log u) \leq \frac{d}{2t} \quad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{1}$$

Since $\partial_t(\log u) - \Delta(\log u) = |\nabla(\log u)|^2$, this is equivalent to

$$\partial_t(\log u) \geq |\nabla(\log u)|^2 - \frac{d}{2t} \quad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{2}$$

- This extends to complete d-dimensional Riemannian manifolds M with $Ric(M) \geq 0$ (Li, Yau, Acta Math. 1986).
- (2) is sharp, one has equality for $u(t,x) = (4\pi t)^{-d/2} \exp\left(\frac{-|x|^2}{4t}\right)$.
- Integration of (2) over a path connecting (t_1, x_1) and (t_2, x_2) with $0 < t_1 < t_2$ gives the sharp Harnack estimate

$$u(t_1, x_1) \leq u(t_2, x_2) \left(\frac{t_2}{t_1}\right)^{d/2} \exp\left(\frac{|x_1 - x_2|^2}{4(t_2 - t_1)}\right).$$

How can we prove Li-Yau? Basic idea: $v := \log u$ solves

$$\partial_t v - \Delta v = |\nabla v|^2, \tag{3}$$

by the chain rule $\Delta H(u) = H'(u)\Delta u + H''(u)|\nabla u|^2$ with $H = \log$.
So we need $-\Delta v \leq \frac{d}{2t}$.

How can we prove Li-Yau? Basic idea: $v := \log u$ solves

$$\partial_t v - \Delta v = |\nabla v|^2, \tag{3}$$

by the chain rule $\Delta H(u) = H'(u)\Delta u + H''(u)|\nabla u|^2$ with $H = \log$. So we need $-\Delta v \leq \frac{d}{2t}$. Apply Δ to (3) and use Bochner's identity.

$$\begin{aligned}\partial_t \Delta v - \Delta(\Delta v) &= \Delta\big(|\nabla v|^2\big) \\ &= 2\nabla v \cdot \nabla \Delta v + 2|\nabla^2 v|_{HS}^2 \quad \Big(+ 2Ric(\nabla v, \nabla v)\Big).\end{aligned}$$

Now, $|\nabla^2 v|_{HS}^2 \geq \frac{1}{d}(\Delta v)^2$ (a CD-inequality), and thus

$$\partial_t \Delta v - \Delta(\Delta v) \geq 2\nabla v \cdot \nabla \Delta v + \frac{2}{d}(\Delta v)^2$$

$\omega(t) := -\frac{d}{2t}$ solves $\partial_t \omega = \frac{2}{d}\omega^2$. Comparison arg. $\hookrightarrow \Delta v \geq \omega$

Γ-calculus by Bakry and Émery

Let L be the generator of a Markov semigroup.

$$\Gamma(f,g) = \frac{1}{2}\big(L(fg) - fLg - gLf\big), \text{ (carré du champ)}$$
$$\Gamma_2(f,g) = \frac{1}{2}\big(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big), \text{ (iterated carré du champ)}$$
$$\Gamma(f) = \Gamma(f,f), \quad \Gamma_2(f) = \Gamma_2(f,f).$$

Γ-calculus by Bakry and Émery

Let L be the generator of a Markov semigroup.

$$\Gamma(f,g) = \frac{1}{2}\big(L(fg) - fLg - gLf\big), \text{ (carré du champ)}$$
$$\Gamma_2(f,g) = \frac{1}{2}\big(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big), \text{ (iterated carré du champ)}$$
$$\Gamma(f) = \Gamma(f,f), \quad \Gamma_2(f) = \Gamma_2(f,f).$$

L satisfies the curvature-dimension inequality $CD(\kappa, d)$ with $\kappa \in \mathbb{R}$ and $d \in [1,\infty]$ if (μ is a fixed invariant and reversible measure)

$$\Gamma_2(f) \geq \kappa\Gamma(f) + \frac{1}{d}(Lf)^2, \quad \mu - a.e.$$

$L = \Delta$ on $\mathbb{R}^d$: $\Gamma(f,g) = \nabla f \cdot \nabla g$, $\Gamma_2(f) = |\nabla^2 f|_{HS}^2$. $\Rightarrow CD(0,d)$

Suppose that L is a diffusion generator, i.e. for all $H \in C^2$ and suitable f the chain rule $LH(f) = H'(f)Lf + H''(f)\Gamma(f)$ holds.

Suppose that L is a diffusion generator, i.e. for all $H \in C^2$ and suitable f the chain rule $LH(f) = H'(f)Lf + H''(f)\Gamma(f)$ holds.

- $CD(0,d)$, $d < \infty$, f pos. $\Rightarrow -L(\log e^{Lt}f) \leq \frac{d}{2t}$, $t > 0$ (Li-Yau)

Suppose that L is a diffusion generator, i.e. for all $H \in C^2$ and suitable f the chain rule $LH(f) = H'(f)Lf + H''(f)\Gamma(f)$ holds.

- $CD(0, d)$, $d < \infty$, f pos. $\Rightarrow -L(\log e^{Lt}f) \leq \frac{d}{2t},\ t > 0$ (Li-Yau)
- $CD(\kappa, \infty)$, $\kappa > 0 \Rightarrow$ log-Sobolev and Poincaré inequality
($\Rightarrow$ exp. decay of entropy and variance, hypercontractivity,...)

See the monograph by Bakry, Gentil, Ledoux, *Analysis and geometry of Markov diffusion operators* (2014)

2. Nonlocal operators: examples and difficulties

Consider Markov generators of the form (X a metric space)

$$Lf(x) = \int_X \big(f(y) - f(x)\big)\, k(x, dy).$$

Key feature: difference structure under integral.

Consider Markov generators of the form (X a metric space)

$$Lf(x) = \int_X \big(f(y) - f(x)\big)\, k(x, dy).$$

Key feature: difference structure under integral. Examples:

- Fractional Laplacian: $X = \mathbb{R}^d$, $\beta \in (0, 2)$,

$$Lf(x) = -\big(-\Delta\big)^{\frac{\beta}{2}} f(x) = c_{\beta,d}\ \mathrm{p.v.} \int_{\mathbb{R}^d} \frac{f(y) - f(x)}{|x-y|^{d+\beta}}\, dy.$$

Consider Markov generators of the form (X a metric space)

$$Lf(x) = \int_X \big(f(y) - f(x)\big)\, k(x, dy).$$

Key feature: difference structure under integral. Examples:

- Fractional Laplacian: $X = \mathbb{R}^d$, $\beta \in (0, 2)$,

$$Lf(x) = -\big(-\Delta\big)^{\frac{\beta}{2}} f(x) = c_{\beta,d}\ \mathrm{p.v.} \int_{\mathbb{R}^d} \frac{f(y) - f(x)}{|x - y|^{d+\beta}}\, dy.$$

- Generators of Markov chains (generalised Laplacians on a graph):

X is a finite or countable set,

$$Lf(x) = \sum_{y \in X} k(x, y)\big(f(y) - f(x)\big).$$

Consider positive solutions u of

$$\partial_t u - Lu = 0, \quad t > 0.$$

Questions: Under which conditions does a Li-Yau inequality hold? Does the Li-Yau estimate imply a parabolic Harnack inequality?

Consider positive solutions u of

$$\partial_t u - Lu = 0, \quad t > 0.$$

Questions: Under which conditions does a Li-Yau inequality hold? Does the Li-Yau estimate imply a parabolic Harnack inequality?

(for $L = -(-\Delta)^{\frac{\beta}{2}}$ addressed in a survey by Garofalo, *Fractional thoughts...* (2019), as a key open problem)

Consider positive solutions u of

$$\partial_t u - Lu = 0, \quad t > 0.$$

Questions: Under which conditions does a Li-Yau inequality hold? Does the Li-Yau estimate imply a parabolic Harnack inequality?

(for $L = -(-\Delta)^{\frac{\beta}{2}}$ addressed in a survey by Garofalo, *Fractional thoughts...* (2019), as a key open problem)

Difficulties:

- The chain rule fails for L.

Consider positive solutions u of

$$\partial_t u - Lu = 0, \quad t > 0.$$

Questions: Under which conditions does a Li-Yau inequality hold? Does the Li-Yau estimate imply a parabolic Harnack inequality?

(for $L = -(-\Delta)^{\frac{\beta}{2}}$ addressed in a survey by Garofalo, *Fractional thoughts...* (2019), as a key open problem)

Difficulties:

- The chain rule fails for L.
- What is the right form of the Li-Yau inequality?

Consider positive solutions u of

$$\partial_t u - Lu = 0, \quad t > 0.$$

Questions: Under which conditions does a Li-Yau inequality hold? Does the Li-Yau estimate imply a parabolic Harnack inequality?

(for $L = -(-\Delta)^{\frac{\beta}{2}}$ addressed in a survey by Garofalo, *Fractional thoughts...* (2019), as a key open problem)

Difficulties:

- The chain rule fails for L.
- What is the right form of the Li-Yau inequality?
- $CD(0, d)$ is not strong enough. New CD-conditions required.

Consider positive solutions u of

$$\partial_t u - Lu = 0, \quad t > 0.$$

Questions: Under which conditions does a Li-Yau inequality hold? Does the Li-Yau estimate imply a parabolic Harnack inequality?

(for $L = -(-\Delta)^{\frac{\beta}{2}}$ addressed in a survey by Garofalo, *Fractional thoughts...* (2019), as a key open problem)

Difficulties:

- The chain rule fails for L.
- What is the right form of the Li-Yau inequality?
- $CD(0, d)$ is not strong enough. New CD-conditions required.
- To obtain Harnack, we have to argue with a "nonlocal gradient".

Positive results on finite and locally finite graphs

• Bauer, Horn, Lin, Lippner, Mangoubi, Yau: Li-Yau inequality on graphs. J. Diff. Geom., 2015:

square root approach, $CDE(\kappa, d)$ and $CDE'(\kappa, d)$, Li-Yau (with $\frac{C}{t}$) and Harnack on finite and infinite graphs

• Münch: Li-Yau inequality on finite graphs via non-linear curvature dimension condition. J. Math. Pures Appl., 2018:

Γ^{ψ}-calculus for concave ψ, $CD_{\psi}(\kappa, d)$, $\psi = \sqrt{}$ covers some of the previous results, $\psi = \log$ gives log. Li-Yau, Harnack; finite graphs

• Dier, Kassmann, Z.: Discrete versions of the Li-Yau gradient estimate. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2021:

log. Li-Yau and Harnack based on $CD(F; 0)$, F a CD-function, more general relaxation functions, improved estimates, some even sharp

On discrete curvature: Erbar, Maas (ARMA 2012), Mielke (Calc. Var. PDE 2013); different approach based on optimal transport

CD-functions

Crucial idea: Replace $(\dots)^2$ in $CD(0,d)$ by a more general function.

Definition

A continuous function $F:[0,\infty)\to[0,\infty)$ is called CD-function, if $F(0)=0$, $F(x)/x$ is strictly increasing on $(0,\infty)$, and $1/F$ is integrable at ∞. (Example: $F(x)=\nu x^2$ with $\nu>0$.)

Lemma

Let $F:[0,\infty)\to[0,\infty)$ be a CD-function. Then there is a unique strictly positive solution φ of the ODE

$$\dot{\varphi}(t)+F(\varphi(t))=0,\quad t>0,$$

which has $(0,\infty)$ as its maximal interval of existence. This function φ is strictly decreasing and log-convex, and it satisfies $\varphi(0+)=\infty$ and $\varphi(\infty)=0$. Call φ the relaxation function associated with F.

Equation for $\log u$

Observation in Dier, Kassmann, Z. (2021): Consider Markov chain setting, i.e.

$$Lf(x) = \sum_{y\in X} k(x,y)\big(f(y) - f(x)\big).$$

Suppose u is positive and $\partial_t u - Lu = 0$ in $(0,\infty)\times X$. Then

$v := \log u$ solves

$$\partial_t v - Lv = \Psi_\Upsilon(v) \quad \text{in } (0,\infty)\times X.$$

$\Psi_H(f)(x) := \sum_{y\in X} k(x,y)H\big(f(y) - f(x)\big)$, $\Upsilon(z) = e^z - 1 - z$

classical case: $\partial_t v - \Delta v = |\nabla v|^2 = \Gamma(v)$,
here $\Gamma(f)(x) = \frac{1}{2}\sum_{y\in X} k(x,y)\big(f(y) - f(x)\big)^2$

3. The CD_Υ condition

Markov chain setting, X finite or countable,

$$Lf(x) = \sum_{y\in X} k(x,y)\big(f(y)-f(x)\big),$$

where $k(x,y)\geq 0$ for $x\neq y$, and $\sum_{y\in X} k(x,y)=0$ for all $x\in X$.

Question: Is there a natural analogue to $CD(\kappa,d)$ with corresponding implications? (Li-Yau, functional inequalities,...)

Markov chain setting, X finite or countable,

$$Lf(x) = \sum_{y\in X} k(x,y)\big(f(y) - f(x)\big),$$

where $k(x,y) \geq 0$ for $x \neq y$, and $\sum_{y\in X} k(x,y) = 0$ for all $x \in X$.

Question: Is there a natural analogue to $CD(\kappa, d)$ with corresponding implications? (Li-Yau, functional inequalities,...)

Recall: $CD(\kappa, d)$ means $\Gamma_2(f) \geq \kappa\Gamma(f) + \frac{1}{d}(Lf)^2$.

Markov chain setting, X finite or countable,

$$Lf(x) = \sum_{y\in X} k(x,y)\big(f(y)-f(x)\big),$$

where $k(x,y)\geq 0$ for $x\neq y$, and $\sum_{y\in X} k(x,y)=0$ for all $x\in X$.

Question: Is there a natural analogue to $CD(\kappa,d)$ with corresponding implications? (Li-Yau, functional inequalities,...)

Recall: $CD(\kappa,d)$ means $\Gamma_2(f)\geq \kappa\Gamma(f)+\frac{1}{d}(Lf)^2$.

We have to identify suitable replacements for all three terms.

In the diffusion setting(chain rule!) with invariant and reversible probability measure μ:

$$\mathcal{H}(P_t f) = \int_X P_t f \log(P_t f)\, d\mu, \quad \text{(entropy)}$$

$$\frac{d}{dt}\mathcal{H}(P_t f) = -\int_X P_t f \Gamma\big(\log(P_t f)\big)\, d\mu, \quad \text{(neg. Fisher information)}$$

$$\frac{d^2}{dt^2}\mathcal{H}(P_t f) = 2\int_X P_t f \Gamma_2\big(\log(P_t f)\big)\, d\mu.$$

In the diffusion setting(chain rule!) with invariant and reversible probability measure μ:

$$\mathcal{H}(P_tf) = \int_X P_tf \log(P_tf)\, d\mu, \quad \text{(entropy)}$$

$$\frac{d}{dt}\mathcal{H}(P_tf) = -\int_X P_tf\Gamma\big(\log(P_tf)\big)\, d\mu, \quad \text{(neg. Fisher information)}$$

$$\frac{d^2}{dt^2}\mathcal{H}(P_tf) = 2\int_X P_tf\Gamma_2\big(\log(P_tf)\big)\, d\mu.$$

In the discrete setting: (see Weber, Z. JFA, 2021; Weber Electron. J. Probab., 2021)

$$\frac{d}{dt}\mathcal{H}(P_tf) = -\int_X P_tf\Psi_\Upsilon\big(\log(P_tf)\big)\, d\mu,$$

$$\frac{d^2}{dt^2}\mathcal{H}(P_tf) = 2\int_X P_tf\Psi_{2,\Upsilon}\big(\log(P_tf)\big)\, d\mu.$$

Here

$$\Psi_{2,\Upsilon}(f) = \frac{1}{2}\big(L\Psi_\Upsilon(f) - B_{\Upsilon'}(f, Lf)\big),$$

where ($\Upsilon'(z) = e^z - 1$)

$$B_{\Upsilon'}(f,g)(x) = \sum_{y\in X} k(x,y)\Upsilon'\big(f(y)-f(x)\big)\big(g(y)-g(x)\big).$$

Here

$$\Psi_{2,\Upsilon}(f) = \frac{1}{2}\big(L\Psi_\Upsilon(f) - B_{\Upsilon'}(f, Lf)\big),$$

where ($\Upsilon'(z) = e^z - 1$)

$$B_{\Upsilon'}(f,g)(x) = \sum_{y\in X} k(x,y)\Upsilon'\big(f(y)-f(x)\big)\big(g(y)-g(x)\big).$$

Definition

L satisfies $CD_\Upsilon(\kappa, F)$ with $\kappa \in \mathbb{R}$ and CD-function F if for all (suitable) $f : X \to \mathbb{R}$ (and with $F_0 := F\chi_{[0,\infty)}$)

$$\Psi_{2,\Upsilon}(f) \geq \kappa\Psi_\Upsilon(f) + F_0(-Lf) \quad \text{on } X.$$

L satisfies $CD_\Upsilon(\kappa, d)$ with $\kappa \in \mathbb{R}$ and $d \in [1,\infty]$ if

$$\Psi_{2,\Upsilon}(f) \geq \kappa\Psi_\Upsilon(f) + \frac{1}{d}(Lf)^2 \quad \text{on } X.$$

Important properties (see Weber, Z. in JFA, 2021):

• $CD_\Upsilon(\kappa, \infty)$ with $\kappa > 0$ implies the modified log-Sobolev inequality (MLSI) with constant κ ($\Rightarrow$ exp. decay of entropy)

MLSI(α) with $\alpha > 0$: $\mathcal{H}(f) \leq \frac{1}{2\alpha}\mathcal{I}(f)$ for all $f \in \mathcal{P}(X)$,

where (with $d\mu = \pi d\#$)

$\mathcal{P}(X) = \{\rho : X \to [0, \infty) \text{ s.t. } \int_X \rho \, d\mu = \sum_{x\in X} \rho(x)\pi(x) = 1\}$,
$\mathcal{H}(f) = \int_X f \log f \, d\mu = \sum_{x\in X} f(x) \log(f(x)) \, \pi(x)$,

Important properties (see Weber, Z. in JFA, 2021):

• $CD_\Upsilon(\kappa,\infty)$ with $\kappa>0$ implies the modified log-Sobolev inequality (MLSI) with constant κ ($\Rightarrow$ exp. decay of entropy)

MLSI(α) with $\alpha>0$: $\mathcal{H}(f)\le\frac{1}{2\alpha}\mathcal{I}(f)$ for all $f\in\mathcal{P}(X)$,

where (with $d\mu=\pi d\#$)

$\mathcal{P}(X)=\{\rho:\ X\to[0,\infty)\text{ s.t. }\int_X\rho\,d\mu=\sum_{x\in X}\rho(x)\pi(x)=1\}$,
$\mathcal{H}(f)=\int_X f\log f\,d\mu=\sum_{x\in X}f(x)\log(f(x))\,\pi(x)$,

and (using detailed balance, i.e. $\pi(x)k(x,y)=\pi(y)k(y,x)$)

$$\begin{aligned}\mathcal{I}(f)&=\int_X f\Psi_\Upsilon(\log f)\,d\mu\\&=\sum_{x\in X}f(x)\sum_{y\in X}k(x,y)\Upsilon\big(\log f(y)-\log f(x)\big)\pi(x)\\&=\frac{1}{2}\sum_{x,y\in X}k(x,y)\big(f(y)-f(x)\big)\big(\log f(y)-\log f(x)\big)\pi(x).\end{aligned}$$

- $CD_\Upsilon(\kappa,\infty)$ is characterised by the gradient bound

$$\Psi_\Upsilon(P_t f) \le e^{-2\kappa t} P_t\big(\Psi_\Upsilon(f)\big), \quad t > 0.$$

- $CD_\Upsilon(\kappa,\infty)$ is characterised by the gradient bound

$$\Psi_\Upsilon(P_t f) \le e^{-2\kappa t} P_t\big(\Psi_\Upsilon(f)\big), \quad t>0.$$

- curvature bounds are preserved under tensorization

L_i: generator of a Markov chain on X_i, $i=1,2$; $L := L_1 \oplus L_2$ generates a chain on $X_1 \times X_2$.

If L_i satisfies $CD_\Upsilon(\kappa_i,\infty)$ with $\kappa_i \in \mathbb{R}$, $i=1,2$, then L satisfies $CD_\Upsilon(\kappa,\infty)$ with $\kappa = \min\{\kappa_1,\kappa_2\}$.

- $CD_\Upsilon(\kappa,\infty)$ is characterised by the gradient bound

$$\Psi_\Upsilon(P_t f) \le e^{-2\kappa t} P_t\big(\Psi_\Upsilon(f)\big), \quad t>0.$$

- curvature bounds are preserved under tensorization

L_i: generator of a Markov chain on X_i, $i=1,2$; $L := L_1 \oplus L_2$ generates a chain on $X_1 \times X_2$.

If L_i satisfies $CD_\Upsilon(\kappa_i,\infty)$ with $\kappa_i \in \mathbb{R}$, $i=1,2$, then L satisfies $CD_\Upsilon(\kappa,\infty)$ with $\kappa = \min\{\kappa_1,\kappa_2\}$.

- CD_Υ is compatible with the diffusion setting; there is a natural notion of $\Gamma \oplus \Psi_\Upsilon$ and $(\Gamma \oplus \Psi_\Upsilon)_2$, tensorization principle for hybrid processes

Further remarks:

- $L\Psi_{\Upsilon'}(f) = 2\Psi_{2,\Upsilon}(f) + B_{exp}(f, Lf)$ with $B_{exp}(f, Lf) \geq 0$ in the situation of the CD condition from [DKZ]

- direct link between $CD_\Upsilon(\kappa, \infty)$ and entropic curvature bound not known; $CD_\Upsilon(\kappa, d)$ is equivalent to Münch's $CD\psi(d, \kappa)$ with $\psi = \log$ (not obvious!); $CD_\Upsilon(\kappa, \infty)$ is closely linked to Monmarché's generalised Γ-calculus (2019)

- $CD_\Upsilon(\kappa, d) \Rightarrow CD(\kappa, d)$.

Examples:

- unweighted complete graph K_n: $CD_\Upsilon(\sqrt{2n},\infty)$ holds for all $n \geq 2$; this is optimal for $n = 2$ since $CD(2,\infty)$ is optimal; $CD(1+\frac{n}{2},\infty)$ is optimal for $n \geq 2$

Examples:

- unweighted complete graph K_n: $CD_\Upsilon(\sqrt{2n}, \infty)$ holds for all $n \geq 2$; this is optimal for $n = 2$ since $CD(2, \infty)$ is optimal; $CD(1 + \frac{n}{2}, \infty)$ is optimal for $n \geq 2$
- n-dimensional hypercube: $CD_\Upsilon(2, \infty)$ holds for all $n \in \mathbb{N}$ and is optimal (since $CD(2, \infty)$ is optimal for all $n \in \mathbb{N}$); easy proof by induction and tensorization and the result for K_2

Examples:

• unweighted complete graph K_n: $CD_\Upsilon(\sqrt{2n},\infty)$ holds for all $n\geq 2$; this is optimal for $n=2$ since $CD(2,\infty)$ is optimal; $CD(1+\frac{n}{2},\infty)$ is optimal for $n\geq 2$

• n-dimensional hypercube: $CD_\Upsilon(2,\infty)$ holds for all $n\in\mathbb{N}$ and is optimal (since $CD(2,\infty)$ is optimal for all $n\in\mathbb{N}$); easy proof by induction and tensorization and the result for K_2

• unweighted 3-star: $CD_\Upsilon(0,\infty)$ fails at center point, but there holds $CD_\Upsilon(\kappa,\infty)$ with some $\kappa\in(-\infty,0)$

4. Long-range jump operators

Consider nonlocal operators on the lattice $\mathbb{Z}$ of the form

$$Lf(x) = \sum_{j\in\mathbb{Z}} k(j)\big(f(x+j) - f(x)\big), \quad x \in \mathbb{Z}, \tag{4}$$

with a kernel $k : \mathbb{Z} \to [0, \infty)$ satisfying $0 < \sum_{j\in\mathbb{Z}} k(j) < \infty$, $k(-j) = k(j)$ for all $j \in \mathbb{N}$, and $k(0) = 0$.

Important example: (linked to frac. discr. Laplacian for $\beta \in (0, 2)$)

$$k_\beta(j) = \frac{c}{|j|^{1+\beta}}, \quad j \in \mathbb{Z} \setminus \{0\}, \quad \text{with } c,\ \beta > 0.$$

Consider nonlocal operators on the lattice $\mathbb{Z}$ of the form

$$Lf(x) = \sum_{j\in\mathbb{Z}} k(j)\big(f(x+j) - f(x)\big), \quad x \in \mathbb{Z}, \tag{4}$$

with a kernel $k : \mathbb{Z} \to [0, \infty)$ satisfying $0 < \sum_{j\in\mathbb{Z}} k(j) < \infty$, $k(-j) = k(j)$ for all $j \in \mathbb{N}$, and $k(0) = 0$.

Important example: (linked to frac. discr. Laplacian for $\beta \in (0, 2)$)

$$k_\beta(j) = \frac{c}{|j|^{1+\beta}}, \quad j \in \mathbb{Z} \setminus \{0\}, \quad \text{with } c, \beta > 0.$$

Theorem: (Spener, Weber, Z., Calc. Var. PDE 2019)

- *k non-incr. on $\mathbb{N}$, $\sum_{j\in\mathbb{N}} k(j) j^2 < \infty \Rightarrow \exists d < \infty : CD(0, d)$ holds.*
- *$k = k_\beta$, $\beta > 2 \Rightarrow CD(0, d)$ holds for some $d < \infty$.*
- *$k = k_\beta$, $\beta < 2 \Rightarrow CD(0, d)$ fails for all $d < \infty$.*

More flexibility by $CD_\Upsilon(0, F)$

What about $\Psi_{2,\Upsilon}(f) \geq F_0(-Lf)$? Note that

$$\Psi_{2,\Upsilon}(f)(x) = \frac{1}{2}\sum_{j,l\in\mathbb{Z}} k(j)k(l)e^{f(x+l)-f(x)}\Upsilon\big(f(x+j+l)-f(x+j)-f(x+l)+f(x)\big),$$

where $\Upsilon(z) = e^z - 1 - z$.

More flexibility by $CD_\Upsilon(0, F)$

What about $\Psi_{2,\Upsilon}(f) \geq F_0(-Lf)$? Note that

$$\Psi_{2,\Upsilon}(f)(x) = \frac{1}{2}\sum_{j,l\in\mathbb{Z}} k(j)k(l)e^{f(x+l)-f(x)}\Upsilon\big(f(x+j+l)-f(x+j)-f(x+l)+f(x)\big),$$

where $\Upsilon(z) = e^z - 1 - z$.

Theorem: (Spener, Weber, Z., DCDS 2024)

- $\sum_{j\in\mathbb{N}} k(j)^{1-\delta} < \infty$, $\delta \in (0,1) \Rightarrow \Psi_{2,\Upsilon}(f) \geq c|Lf|^\gamma$, $\gamma = \frac{1+\delta}{\delta}$.

More flexibility by $CD_\Upsilon(0, F)$

What about $\Psi_{2,\Upsilon}(f) \geq F_0(-Lf)$? Note that

$$\Psi_{2,\Upsilon}(f)(x) = \frac{1}{2}\sum_{j,l\in\mathbb{Z}} k(j)k(l)e^{f(x+l)-f(x)}\Upsilon\big(f(x+j+l)-f(x+j)-f(x+l)+f(x)\big),$$

where $\Upsilon(z) = e^z - 1 - z$.

Theorem: (Spener, Weber, Z., DCDS 2024)

- $\sum_{j\in\mathbb{N}} k(j)^{1-\delta} < \infty$, $\delta \in (0,1) \Rightarrow \Psi_{2,\Upsilon}(f) \geq c|Lf|^\gamma$, $\gamma = \frac{1+\delta}{\delta}$.
- $k = k_\beta$, $\beta \in (0,\infty) \Rightarrow CD_\Upsilon(0, F)$ holds with some CD-function F, which grows exponentially at ∞ and satisfies $F(x) \sim cx^\gamma$ as $x \to 0$ where $\gamma = 2$ for $\beta > 2$ and ($\beta_* = \frac{1+\sqrt{5}}{2}$)

$$\gamma = \frac{1+2\beta}{\beta} \text{ for } \beta \in (0, \beta_*],\ \gamma = \frac{\beta-\varepsilon}{\beta-\varepsilon-1} \text{ for } \beta \in (\beta_*, 2].$$

Theorem: (Spener, Weber, Z., DCDS 2024)

Let $\beta \in (0,\infty)$ and L_β the operator associated with k_β. Any bdd. function $u : [0,\infty) \times \mathbb{Z} \to (0,\infty)$ that is C^1 in time and solves $\partial_t u = L_\beta u$ on $(0,\infty) \times \mathbb{Z}$ satisfies the Li-Yau estimate

$$-L_\beta \log u(t,x) \le \varphi(t), \quad (t,x) \in (0,\infty) \times \mathbb{Z},$$

where φ is the relax. function corresp. to F from above.
($\varphi(t) \sim -c \log t$ as $t \to 0$ and $\varphi(t) \sim ct^{-\frac{1}{\gamma-1}}$ as $t \to \infty$)
Moreover, we have the Harnack inequality

$$u(t_1,x_1) \le u(t_2,x_2) \exp\Big(\int_{t_1}^{t_2} \varphi(t)\,\mathrm{d}t + \frac{2|x_1-x_2|^{\min\{1+\beta,2\}}}{t_2-t_1} \Big),$$

for $0 \le t_1 < t_2$, $x_1, x_2 \in \mathbb{Z}$. ($t_1 = 0$ possible, φ is integrable at 0!)

Theorem: (Spener, Weber, Z., DCDS 2024)

Let $\beta \in (0,\infty)$ and L_β the operator associated with k_β. Any bdd. function $u : [0,\infty) \times \mathbb{Z} \to (0,\infty)$ that is C^1 in time and solves $\partial_t u = L_\beta u$ on $(0,\infty) \times \mathbb{Z}$ satisfies the Li-Yau estimate

$$-L_\beta \log u(t,x) \leq \varphi(t), \quad (t,x) \in (0,\infty) \times \mathbb{Z},$$

where φ is the relax. function corresp. to F from above.
$(\varphi(t) \sim -c \log t$ as $t \to 0$ and $\varphi(t) \sim ct^{-\frac{1}{\gamma-1}}$ as $t \to \infty)$
Moreover, we have the Harnack inequality

$$u(t_1,x_1) \leq u(t_2,x_2) \exp\Big(\int_{t_1}^{t_2} \varphi(t)\, \mathrm{d}t + \frac{2|x_1 - x_2|^{\min\{1+\beta,2\}}}{t_2 - t_1} \Big),$$

for $0 \leq t_1 < t_2$, $x_1, x_2 \in \mathbb{Z}$. ($t_1 = 0$ possible, φ is integrable at 0!)

Remark: Same result for frac. discrete Laplacian $-(-\Delta)^{\frac{\beta}{2}}$ & heat kernel estimates

The fractional Laplacian has infinite dimension

Let $\beta \in (0,2)$ and consider for $x \in \mathbb{R}^d$

$$Lu(x) = -(-\Delta)^{\frac{\beta}{2}} u(x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{u(x+h) - 2u(x) + u(x-h)}{|h|^{d+\beta}} \, dh.$$

$$\Gamma(u)(x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{(u(x+h) - u(x))^2}{|h|^{d+\beta}} \, dh,$$

$$\Gamma_2(u)(x) = c_{\beta,d}^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{[u(x+h+\sigma) - u(x+h) - u(x+\sigma) + u(x)]^2}{|h|^{d+\beta}|\sigma|^{d+\beta}} \, dh \, d\sigma.$$

The fractional Laplacian has infinite dimension

Let $\beta \in (0,2)$ and consider for $x \in \mathbb{R}^d$

$$Lu(x) = -(-\Delta)^{\frac{\beta}{2}} u(x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{u(x+h) - 2u(x) + u(x-h)}{|h|^{d+\beta}} \, dh.$$

$$\Gamma(u)(x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{(u(x+h) - u(x))^2}{|h|^{d+\beta}} \, dh,$$

$$\Gamma_2(u)(x) = c_{\beta,d}^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{[u(x+h+\sigma) - u(x+h) - u(x+\sigma) + u(x)]^2}{|h|^{d+\beta}|\sigma|^{d+\beta}} \, dh \, d\sigma.$$

Theorem: (Spener, Weber, Z., Comm. PDE 2020)

For any $R > 0$, $\kappa \in \mathbb{R}$ and $N \in (0,\infty)$ there is a $u \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$0 < \Gamma_2(u)(x) < \kappa \Gamma(u)(x) + \frac{1}{N} \left(L(u)(x)\right)^2, \quad \forall x \in B(0,R).$$

The fractional Laplacian has infinite dimension

Let $\beta \in (0,2)$ and consider for $x \in \mathbb{R}^d$

$$Lu(x) = -(-\Delta)^{\frac{\beta}{2}} u(x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{u(x+h) - 2u(x) + u(x-h)}{|h|^{d+\beta}}\, dh.$$

$$\Gamma(u)(x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{(u(x+h) - u(x))^2}{|h|^{d+\beta}}\, dh,$$

$$\Gamma_2(u)(x) = c_{\beta,d}^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{[u(x+h+\sigma) - u(x+h) - u(x+\sigma) + u(x)]^2}{|h|^{d+\beta}|\sigma|^{d+\beta}}\, dh\, d\sigma.$$

Theorem: (Spener, Weber, Z., Comm. PDE 2020)

For any $R > 0$, $\kappa \in \mathbb{R}$ and $N \in (0,\infty)$ there is a $u \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$0 < \Gamma_2(u)(x) < \kappa\Gamma(u)(x) + \frac{1}{N}\left(L(u)(x)\right)^2, \quad \forall x \in B(0,R).$$

(i) negative answer to a question by Garofalo (2019)

(ii) $CD_\Upsilon(0,d)$ must fail as well, but scaling suggests $\varphi(t) \leq \frac{c}{t}$;

thus only little hope w.r.t. the CD approach! Other CD-conditions?

5. Reduction to the heat kernel

Reduction to the heat kernel for the heat equation

Lemma: Let $Pf(x) = \int_{\mathbb{R}^d} H(x,y)f(y)\,dy$, for sufficiently regular, positive functions H and f. Then

$$\int_{\mathbb{R}^d} \left|\nabla_x \log H(x,y)\right|^2 H(x,y)f(y)\,dy \geq \left|\nabla \log Pf(x)\right|^2 Pf(x). \quad (5)$$

Proof: By Hölder's inequality we have

$$\begin{aligned}(\partial_{x_i} Pf(x))^2 &= \Big(\int_{\mathbb{R}^d} \partial_{x_i} H(x,y)f(y)\,dy\Big)^2 \\ &\leq \int_{\mathbb{R}^d} \frac{(\partial_{x_i} H(x,y))^2}{H(x,y)} f(y)\,dy \int_{\mathbb{R}^d} H(x,y)f(y)\,dy,\end{aligned}$$

which directly leads to (5) by summing up and employing the chain rule for the gradient ($\nabla(\log g) = \frac{\nabla g}{g}$).

For the heat kernel $H(t,x,y)=(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}$ we have

$$-\Delta_x\big(\log H(t,x,y)\big)=\frac{d}{2t}=:\varphi(t). \tag{6}$$

For any positive solution u of the heat equation,

$$\partial_t u-u\Delta(\log u)=u|\nabla(\log u)|^2.$$

In particular,

$$\partial_t H(t,x,y)+H(t,x,y)\varphi(t)=H(t,x,y)|\nabla_x(\log H(t,x,y))|^2$$

Consider a positive solution $u(t,x) = \int_{\mathbb{R}^d} H(t,x,y)u_0(y)\,dy$ of the heat equation (Widder's type theorem!). Then

$$\begin{aligned}
\partial_t u(t,x) + \varphi(t)u(t,x) &= \int_{\mathbb{R}^d} \big(\partial_t H(t,x,y) + \varphi(t)H(t,x,y)\big)\,u_0(y)\,dy \\
&= \int_{\mathbb{R}^d} \big(|\nabla_x(\log H(t,x,y))|^2 H(t,x,y)\big)\,u_0(y)\,dy \\
&\geq |\nabla(\log u(t,x))|^2 u(t,x) \qquad \text{(by the Lemma)} \\
&= \partial_t u(t,x) - u(t,x)\Delta(\log u(t,x)).
\end{aligned}$$

Hence

$$-\Delta(\log u(t,x)) \leq \varphi(t) = \frac{d}{2t}.$$

Note that we only need $-\Delta_x\big(\log H(t,x,y)\big) \leq \varphi(t)$.

Conclusion: Li-Yau for heat kernel implies Li-Yau for pos. solutions.

Reduction to the heat kernel for the fractional heat equation

Question: Is there a similar argument for the fractional heat equation (FHE)?

Let u be a positive solution of the FHE. Then

$$\partial_t(\log u) + (-\Delta)^{\frac{\beta}{2}}(\log u) = \Psi_\Upsilon(\log u),$$

where

$$\Psi_\Upsilon(v)(x) = c_{\beta,d} \int_{\mathbb{R}^d} \frac{\Upsilon\big(v(y) - v(x)\big)}{|x-y|^{d+\beta}}\, dy$$

with $\Upsilon(z) = e^z - 1 - z$. Equivalently,

$$\partial_t u + u(-\Delta)^{\frac{\beta}{2}}(\log u) = u\Psi_\Upsilon(\log u).$$

Recall the key inequality from the local case

$$\int_{\mathbb{R}^d} \left|\nabla_x \log H(x,y)\right|^2 H(x,y) f(y)\, dy \geq \left|\nabla \log Pf(x)\right|^2 Pf(x),$$

where $Pf(x) = \int_{\mathbb{R}^d} H(x,y) f(y)\, dy$ and H and f are sufficiently regular, positive functions.

Recall the key inequality from the local case

$$\int_{\mathbb{R}^d} \left|\nabla_x \log H(x,y)\right|^2 H(x,y)f(y)\,dy \geq \left|\nabla \log Pf(x)\right|^2 Pf(x),$$

where $Pf(x) = \int_{\mathbb{R}^d} H(x,y)f(y)\,dy$ and H and f are sufficiently regular, positive functions.

Key lemma: (Weber, Z. in Math. Ann. 2023) Let P, H, f be as before. Then

$$\int_{\mathbb{R}^d} \Psi_\Upsilon(\log H(\cdot,y))(x)H(x,y)f(y)\,dy \geq \Psi_\Upsilon(\log Pf)(x)Pf(x).$$

Proof: Use the convexity of $r \mapsto \Upsilon(\log r) = r - \log r - 1$.

Remark: The lemma extends to more general nonlocal operators.

Positive (strong) solutions u of the FHE can be expressed as

$$u(t,x) = \int_{\mathbb{R}^d} G^{(\beta)}(t, x-y) u_0(y)\, dy, \tag{7}$$

where $G^{(\beta)}$ is the fund. sol. of the FHE, see Barrios, Peral, Soria, Valdinoci, ARMA (2014).

Set $H(t,x,y) = G^{(\beta)}(t,x-y)$. Using the lemma we can argue as before to see the implication

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x) \le \varphi(t) \Rightarrow (-\Delta)^{\frac{\beta}{2}}(\log u)(t,x) \le \varphi(t).$$

Question: For which function φ do we have

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x) \le \varphi(t) \quad ?$$

Lemma: For all $\beta \in (0,2)$, $t > 0$, and $x \in \mathbb{R}^d$, we have

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x) \leq \frac{C_{LY}(\beta,d)}{t},$$

where the finite constant $C_{LY}(\beta,d) > 0$ is given by

$$C_{LY}(\beta,d) = \frac{c_{\beta,d}}{2} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\log\left(\frac{\Phi_\beta(y)^2}{\Phi_\beta(y+\sigma)\Phi_\beta(y-\sigma)}\right)}{|\sigma|^{d+\beta}} \mathrm{d}\sigma,$$

with $\Phi_\beta(y) = G^{(\beta)}(1,y)$, $y \in \mathbb{R}^d$.

Note that $G^{(\beta)}(t,x) = t^{-\frac{d}{\beta}}\Phi_\beta\left(xt^{-\frac{1}{\beta}}\right)$. $C_{LY}(1,d) = \frac{d(d+1)}{2B\left(\frac{d+1}{2},\frac{1}{2}\right)}$.
$C_{LY}(\beta,d)$ is the smallest constant among all $C > 0$ satisfying

$$(-\Delta)^{\frac{\beta}{2}}(\log G^{(\beta)})(t,x) \leq \frac{C}{t}, \quad t > 0,\ x \in \mathbb{R}^d.$$

Theorem: (Weber, Z., Math. Ann. 2023))
Let $\beta \in (0,2)$ and $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ a strong solution of

$$\partial_t u + (-\Delta)^{\frac{\beta}{2}} u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^d.$$

Then for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$, we have the Li-Yau inequality

$$\big(-\Delta\big)^{\frac{\beta}{2}} (\log u)(t,x) \leq \frac{C_{LY}(\beta,d)}{t} \tag{8}$$

and, equivalently, the differential Harnack inequality

$$\partial_t (\log u)(t,x) \geq \Psi_\Upsilon(\log u)(t,x) - \frac{C_{LY}(\beta,d)}{t}. \tag{9}$$

Theorem: (Weber, Z., Math. Ann. 2023))
Let $\beta \in (0,2)$ and $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ a strong solution of

$$\partial_t u + (-\Delta)^{\frac{\beta}{2}} u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^d.$$

Then for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$, we have the Li-Yau inequality

$$\big(-\Delta\big)^{\frac{\beta}{2}}(\log u)(t,x) \leq \frac{C_{LY}(\beta,d)}{t} \tag{8}$$

and, equivalently, the differential Harnack inequality

$$\partial_t(\log u)(t,x) \geq \Psi_{\Upsilon}(\log u)(t,x) - \frac{C_{LY}(\beta,d)}{t}. \tag{9}$$

Remark: different form of differential Harnack inequality with a $|\nabla \log u|^2$ term by Ghosh, Kassmann, unpublished (2020/2021)

Theorem: (Weber, Z., Math. Ann. 2023))
Let $\beta \in (0,2)$ and $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ a strong solution of

$$\partial_t u + (-\Delta)^{\frac{\beta}{2}} u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^d.$$

Then for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$, we have the Li-Yau inequality

$$\big(-\Delta\big)^{\frac{\beta}{2}}(\log u)(t,x) \le \frac{C_{LY}(\beta,d)}{t} \tag{8}$$

and, equivalently, the differential Harnack inequality

$$\partial_t(\log u)(t,x) \ge \Psi_\Upsilon(\log u)(t,x) - \frac{C_{LY}(\beta,d)}{t}. \tag{9}$$

Remark: different form of differential Harnack inequality with a $|\nabla \log u|^2$ term by Ghosh, Kassmann, unpublished (2020/2021)

Question: Can we derive a Harnack inequality from (9)?

Li-Yau implies Harnack for the fractional heat equation

Turns out to be much more involved than in the classical case and in the discrete setting!

Theorem : (Weber, Z. in Math. Ann., 2023)
Let $\beta \in (0,2)$ and $u : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ be sufficiently smooth. Then the differential Harnack inequality

$$\partial_t(\log u)(t,x) \geq \Psi_\Upsilon(\log u)(t,x) - \frac{C_{LY}(\beta,d)}{t}, \; (t,x) \in (0,\infty)\times\mathbb{R}^d,$$

implies that there exists a constant $C = C(\beta,d) > 0$ s.t. for all $0 < t_1 < t_2 < \infty$ and $x_1, x_2 \in \mathbb{R}^d$ there holds

$$u(t_1,x_1) \leq u(t_2,x_2)\Big(\frac{t_2}{t_1}\Big)^{C_{LY}} \exp\left(C\left[1 + \frac{|x_1 - x_2|^{\beta+d}}{(t_2-t_1)^{1+\frac{d}{\beta}}}\right]\right).$$

Some known results on the parabolic Harnack inequality for the space fractional heat equation:

• Bass, Levin, Trans. Amer. Math. Soc. (2002); Chen, Kumagai, Stochastic Process. Appl. (2003): local solutions, probabilistic methods

• Chang-Lara, D'avila, J. Differential Equations (2016): local solutions in a rough non-var. setting, purely analytic proof

• Kassmann, Weidner, arxiv (2024): local solutions in rough variational setting

• Bonforte, Sire, Vázquez, Nonlinear Anal. (2017): global solutions, estimates based on fundamental solution, Harnack inequalities of forward/backward/elliptic type

Improved differential Harnack inequality

The fundamental solution $G^{(\beta)}$, $\beta \in (0,2)$, satisfies

$$t|\partial_t G^{(\beta)}(t,x)| \leq c_{\beta,d} G^{(\beta)}(t,x), \quad t>0,\, x \in \mathbb{R}^d, \tag{10}$$

see Vázquez, de Pablo, Quirós, Rodriguez (JEMS 2017).
Bonforte, Sire, Vázquez (2017) also use (10) and show in addition $t\partial_t G^{(\beta)} \geq -\frac{d}{\beta} G^{(\beta)}$. By means of (10) we can show

Theorem: (Weber, Z., in prep.)
For any strong sol. $u : [0,\infty) \times \mathbb{R}^d \to (0,\infty)$ of FHE ($\beta \in (0,2)$)

$$|\partial_t(\log u)| + \Psi_\Upsilon(\log u) \leq \frac{\tilde{C}(\beta,d)}{t} \text{ in } (0,\infty) \times \mathbb{R}^d. \tag{11}$$

(11) yields Harnack inequalities of forward/backward/elliptic type.

6. Hybrid problems: approach based on CD-conditions

Consider the reaction-diffusion system ($i = 1, \ldots, m$)

$$\partial_t u_i(t,x) - \Delta u_i(t,x) = \sum_{j=1}^{m} k(i,j)\big(u_j(t,x) - u_i(t,x)\big), \quad t > 0,\ x \in \mathbb{R}^n, \tag{12}$$

where k is as in the Markov chain setting. Simplest example:

$$\begin{cases} \partial_t u_1(t,x) - \Delta u_1(t,x) = u_2(t,x) - u_1(t,x), \\ \partial_t u_2(t,x) - \Delta u_2(t,x) = u_1(t,x) - u_2(t,x). \end{cases}$$

special structure of RHS in (12) $\Rightarrow$ cooperative parabolic system, Harnack estimates known (Földes, Poláĉik in DCDS, 2009)

Question: Can we derive differential Harnack inequalities?

Idea: Consider the index $i \in \{1, \ldots, m\}$ as additional spatial variable, define $u(t, x, i) := u_i(t, x)$. Write the system as

$$\partial_t u - \Delta u - L_d u = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n \times \{1, \ldots, m\},$$

where L_d acts in i, $L_d f(i) = \sum_{j=1}^m k(i,j)\big(f(j) - f(i)\big)$.

Idea: Consider the index $i \in \{1, \dots, m\}$ as additional spatial variable, define $u(t, x, i) := u_i(t, x)$. Write the system as

$$\partial_t u - \Delta u - L_d u = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n \times \{1, \dots, m\},$$

where L_d acts in i, $L_d f(i) = \sum_{j=1}^m k(i,j)\big(f(j) - f(i)\big)$.

More generally, consider an operator sum of Markov generators $L_c \oplus L_d$ on $X \times Y$ where:

L_c acts w.r.t. the continuous variable $x \in X$ (with diffusion property),
L_d acts w.r.t. the discrete variable $y \in Y$.

Hybrid CD-condition?

Def.: $L_c \oplus L_d$ satisfies $CD_{hyb}(\kappa, d)$ for some $\kappa \in \mathbb{R}$ and $d \in [1, \infty]$ if for all suitable $f : X \times Y \to \mathbb{R}$

$$(\Gamma \oplus \Psi_\Upsilon)_2 (f) \geq \kappa \big(\Gamma \oplus \Psi_\Upsilon\big)(f) + \frac{1}{d}\big((L_c \oplus L_d)f\big)^2.$$

Here

$$\begin{aligned}(\Gamma \oplus \Psi_\Upsilon)_2 (f) := \frac{1}{2}\Big(&(L_c \oplus L_d)(\Gamma \oplus \Psi_\Upsilon)(f) \\ &- 2\Gamma\big(f, (L_c \oplus L_d)(f)\big) - B_{\Upsilon'}\big(f, (L_c \oplus L_d)(f)\big)\Big)\end{aligned}$$

Def.: $L_c \oplus L_d$ satisfies $CD_{hyb}(\kappa, d)$ for some $\kappa \in \mathbb{R}$ and $d \in [1, \infty]$ if for all suitable $f : X \times Y \to \mathbb{R}$

$$(\Gamma \oplus \Psi_\Upsilon)_2 (f) \geq \kappa \big(\Gamma \oplus \Psi_\Upsilon\big)(f) + \frac{1}{d}\big((L_c \oplus L_d)f\big)^2.$$

Here

$$(\Gamma \oplus \Psi_\Upsilon)_2 (f) := \frac{1}{2}\Big((L_c \oplus L_d)\,(\Gamma \oplus \Psi_\Upsilon)\,(f) \\ - 2\Gamma\big(f, (L_c \oplus L_d)(f)\big) - B_{\Upsilon'}\big(f, (L_c \oplus L_d)(f)\big)\Big)$$

Tensorization principle: (Kräss, Z. on arxiv, 2023)

If L_c satisfies $CD(\kappa_1, d_1)$ and L_d satisfies $CD_\Upsilon(\kappa_2, d_2)$, then $L_c \oplus L_d$ satisfies $CD_{hyb}(\kappa, d)$ with $\kappa = \min\{\kappa_1, \kappa_2\}$, $d = d_1 + d_2$.

Back to the sum $\mathcal{L} = \Delta \oplus L_d$ on $\mathbb{R}^n \times Y$, $Y = \{1, \ldots, m\}$.
Assume: Y connected and $k(y_1, y_2) > 0 \Leftrightarrow k(y_2, y_1) > 0$.
k_0 is minimum of all pos. k values.

Theorem: (Kräss, Z. on arxiv, 2023)

Let $u : [0, \infty) \times \mathbb{R}^n \times Y \to (0, \infty)$ be sufficiently smooth and $\partial_t u - \mathcal{L}u = 0$ on $(0, \infty) \times \mathbb{R}^n \times Y$. Assume $\mathcal{L}$ satisfies $CD_{hyb}(0, d)$ for some $d \in [1, \infty)$. Then $v = \log u$ satisfies

$$-\mathcal{L}v = \big(|\nabla v|^2 + \Psi_\Upsilon(v)\big) - \partial_t v \leq \frac{d}{2t} \quad \text{on } (0, \infty) \times \mathbb{R}^n \times Y.$$

Moreover, for $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in Y$ and $0 < t_1 < t_2 < \infty$,

$$u(t_1, x_1, y_1) \leq u(t_2, x_2, y_2) \Big(\frac{t_2}{t_1}\Big)^{\frac{d}{2}} \exp\Big(\frac{|x_2 - x_1|^2}{4(t_2 - t_1)} + 2\frac{\operatorname{dist}(y_1, y_2)^2}{k_0(t_2 - t_1)}\Big).$$

Back to the sum $\mathcal{L} = \Delta \oplus L_d$ on $\mathbb{R}^n \times Y$, $Y = \{1, \ldots, m\}$.
Assume: Y connected and $k(y_1, y_2) > 0 \Leftrightarrow k(y_2, y_1) > 0$.
k_0 is minimum of all pos. k values.

Theorem: (Kräss, Z. on arxiv, 2023)

Let $u : [0, \infty) \times \mathbb{R}^n \times Y \to (0, \infty)$ be sufficiently smooth and $\partial_t u - \mathcal{L}u = 0$ on $(0, \infty) \times \mathbb{R}^n \times Y$. Assume $\mathcal{L}$ satisfies $CD_{hyb}(0, d)$ for some $d \in [1, \infty)$. Then $v = \log u$ satisfies

$$-\mathcal{L}v = \left(|\nabla v|^2 + \Psi_\Upsilon(v)\right) - \partial_t v \leq \frac{d}{2t} \quad \text{on } (0, \infty) \times \mathbb{R}^n \times Y.$$

Moreover, for $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in Y$ and $0 < t_1 < t_2 < \infty$,

$$u(t_1, x_1, y_1) \leq u(t_2, x_2, y_2) \left(\frac{t_2}{t_1}\right)^{\frac{d}{2}} \exp\left(\frac{|x_2 - x_1|^2}{4(t_2 - t_1)} + 2\frac{\operatorname{dist}(y_1, y_2)^2}{k_0(t_2 - t_1)}\right).$$

Remark: There is also a version for local (w.r.t. x) solutions.

Illustration: Consider the simple RD-system

$$\begin{cases} \partial_t u_1(t,x) - \Delta u_1(t,x) = u_2(t,x) - u_1(t,x), \\ \partial_t u_2(t,x) - \Delta u_2(t,x) = u_1(t,x) - u_2(t,x). \end{cases}$$

Here, Y is the two-point graph, L_d satisfies $CD_\Upsilon(0, d_2)$ with $d_2 \approx 1,258$. Δ satisfies $CD(0,n)$. By tensorization, $\mathcal{L} = \Delta \oplus L_d$ satisfies $CD_{hyb}(0,d)$ with $d = n + d_2$.

Illustration: Consider the simple RD-system

$$\begin{cases} \partial_t u_1(t,x) - \Delta u_1(t,x) = u_2(t,x) - u_1(t,x), \\ \partial_t u_2(t,x) - \Delta u_2(t,x) = u_1(t,x) - u_2(t,x). \end{cases}$$

Here, Y is the two-point graph, L_d satisfies $CD_\Upsilon(0,d_2)$ with $d_2 \approx 1,258$. Δ satisfies $CD(0,n)$. By tensorization, $\mathcal{L} = \Delta \oplus L_d$ satisfies $CD_{hyb}(0,d)$ with $d = n + d_2$. We obtain that for $0 < t_1 < t_2 < \infty$, $x_1, x_2 \in \mathbb{R}^n$, and $i,j \in \{1,2\}$,

$$u_i(t_1,x_1) \le u_j(t_2,x_2)\Big(\frac{t_2}{t_1}\Big)^{\frac{n+d_2}{2}} \exp\Big(\frac{|x_2-x_1|^2}{4(t_2-t_1)} + \frac{2|j-i|^2}{t_2-t_1}\Big).$$

Illustration: Consider the simple RD-system

$$\begin{cases} \partial_t u_1(t,x) - \Delta u_1(t,x) = u_2(t,x) - u_1(t,x), \\ \partial_t u_2(t,x) - \Delta u_2(t,x) = u_1(t,x) - u_2(t,x). \end{cases}$$

Here, Y is the two-point graph, L_d satisfies $CD_\Upsilon(0,d_2)$ with $d_2 \approx 1,258$. Δ satisfies $CD(0,n)$. By tensorization, $\mathcal{L} = \Delta \oplus L_d$ satisfies $CD_{hyb}(0,d)$ with $d = n + d_2$. We obtain that for $0 < t_1 < t_2 < \infty$, $x_1, x_2 \in \mathbb{R}^n$, and $i,j \in \{1,2\}$,

$$u_i(t_1,x_1) \le u_j(t_2,x_2)\Big(\frac{t_2}{t_1}\Big)^{\frac{n+d_2}{2}} \exp\Big(\frac{|x_2-x_1|^2}{4(t_2-t_1)} + \frac{2|j-i|^2}{t_2-t_1}\Big).$$

Remark: Open problem: different diffusion coefficients; then tensorization fails, since the operators in the sum do not commute

THANK YOU FOR YOUR ATTENTION!

References

- A. Spener, F. Weber, R. Zacher: Curvature-dimension inequalities for non-local operators in the discrete setting. Calc. Var. Part. Diff. Equ. **58** (2019), Paper No. 171.
- A. Spener, F. Weber, R. Zacher: The fractional Laplacian has infinite dimension. Comm. Partial Differential Equations **45** (2020), 57–75.
- F. Weber, R. Zacher: The entropy method under curvature-dimension conditions in the spirit of Bakry-Émery in the discrete setting of Markov chains. J. Funct. Anal. 281 (2021), no. 5, Paper No. 109061.
- F. Weber, R. Zacher: Li-Yau inequalities for general non-local diffusion equations via reduction to the heat kernel. Math. Ann. **385** (2023), 393–419.
- S. Kräss, F. Weber, R. Zacher: Li-Yau and Harnack inequalities via curvature-dimension conditions for discrete long-range jump operators including the fractional discrete Laplacian. Discrete Contin. Dyn. Syst. **44** (2024), 1982–2028.