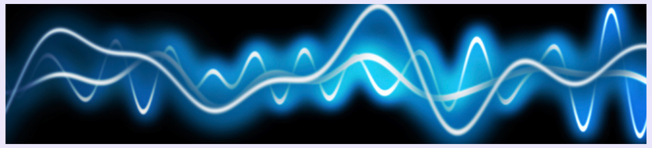


# Wave and heat processes : Some connections

Enrique Zuazua

GHENT UNIVERSITY

## The control problem

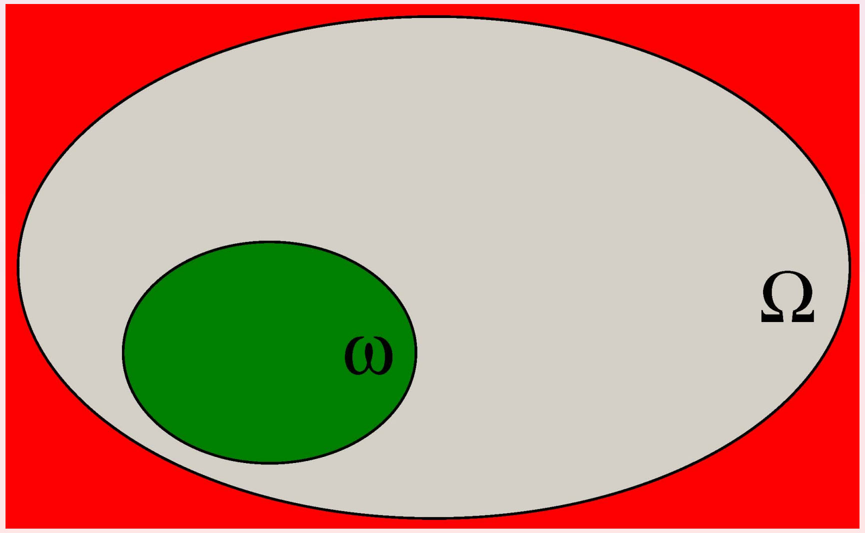
Let $n \geq 1$ and $T > 0$, $\Omega$ be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary $\Gamma$, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

$1_\omega =$ the characteristic function of $\omega$ of $\Omega$ where the control is active.
We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (1) admits an unique solution

$$u \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

$$u = u(x,t) = \textit{solution} = \textit{state},\ f = f(x,t) = \textit{control}$$



Well known result (Fursikov-Imanuvilov, Lebeau-Robbiano,...) : The system is null-controllable in any time $T$ and from any open non-empty subset $\omega$ of $\Omega$.

The control of minimal $L^2$-norm can be found by minimizing

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx \tag{2}$$

over the space of solutions of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(T,x) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{3}$$

Obviously, the functional is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$ and coercive because of the observability estimate:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{4}$$

One has in fact

$$\int_0^T \int_\Omega e^{\frac{-A}{(T-t)}} \varphi^2 dxdt \leq C \int_0^T \int_\omega \varphi^2 dxdt.$$

Open problem # 1.1: Characterize the best constant $A$ in this inequality:

$$A = A(\Omega, \omega).$$

The Carleman inequality approach allows establishing some upper bounds on $A$ depending on the properties of the weight function. But this does not give a clear path towards the obtention of a sharp constant.

## Kannai Transform

The **Kannai transform** allows transferring the results we have obtained for the wave equation to other models and in particular to the heat equation (Y. Kannai, 1977; K. D. Phung, 2001; L. Miller, 2004)

$$e^{t\Delta}\varphi = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} W(s) ds$$

where $W(x, s)$ solves the corresponding wave equation with data $(\varphi, 0)$.

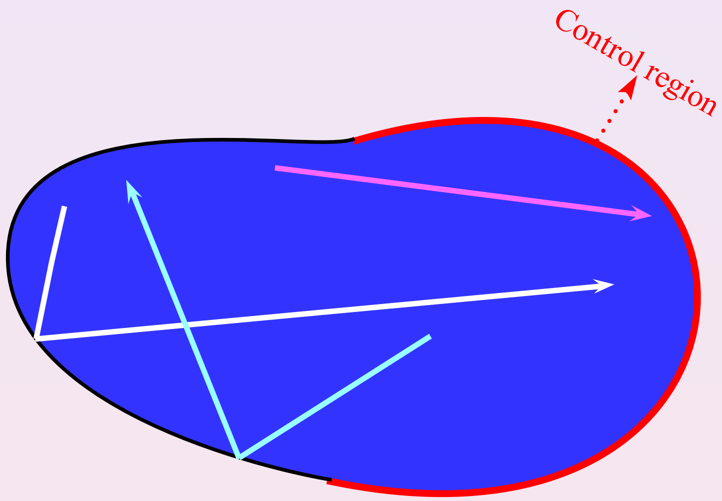
$$W_{ss} + AW = 0 \quad + \quad K_t - K_{ss} = 0 \quad \rightarrow \quad U_t + AU = 0,$$

$$W_{ss} + AW = 0 \quad + \quad iK_t - K_{ss} = 0 \quad \rightarrow \quad iU_t + AU = 0.$$

This can be actually applied in a more general abstract context $(U_t + AU = 0)$ but not when the equation has time-dependent coefficients.

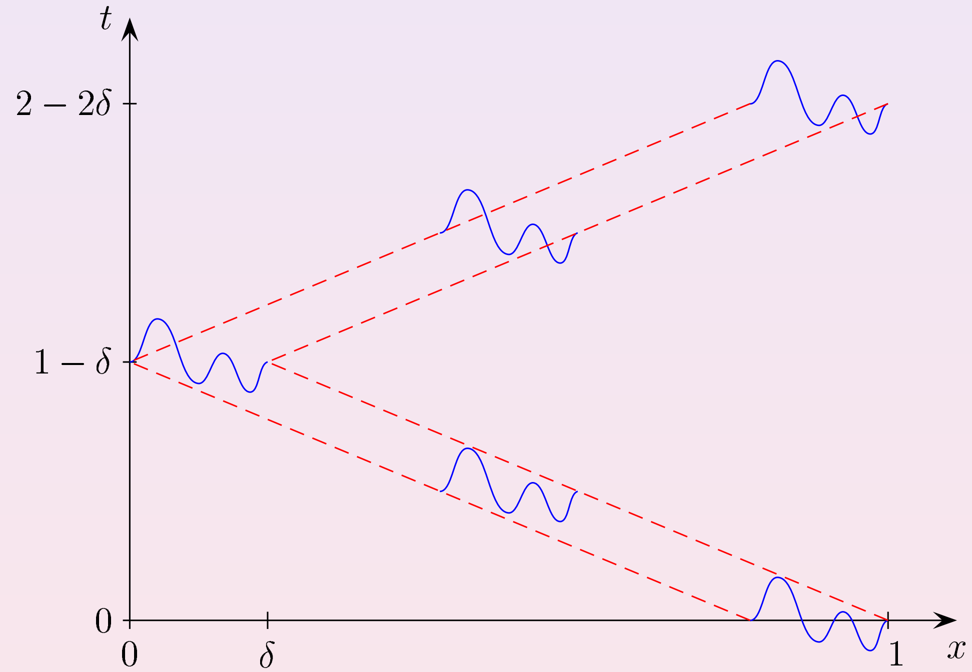
# GCC

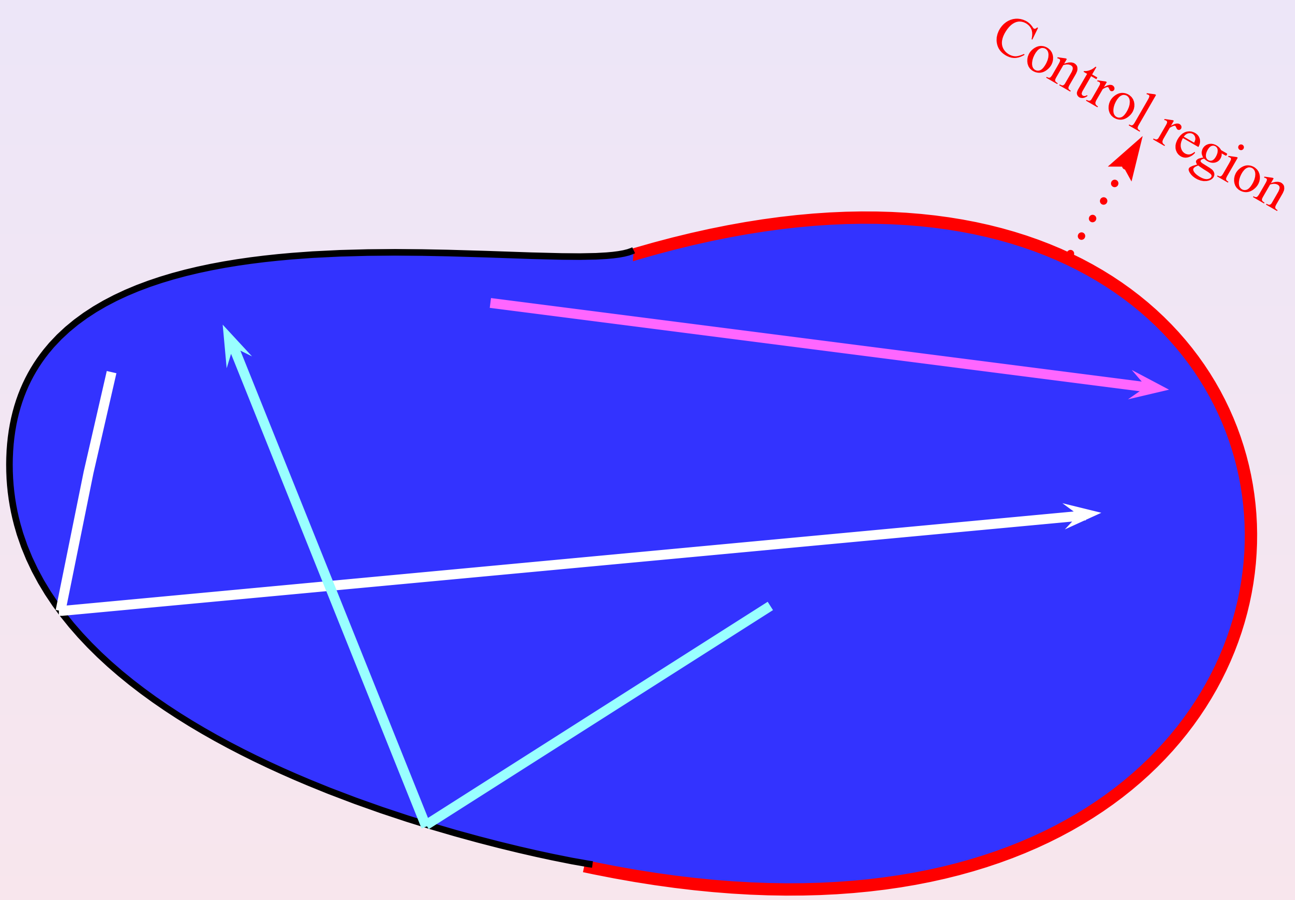
**The observability inequality for waves propagation phenomena holds if and only if the support of the dissipative mechanism, $\Gamma_0$ or $\omega$, satisfies the so called the Geometric Control Condition (GCC)** (Ralston, Rauch-Taylor, Bardos-Lebeau-Rauch,...)



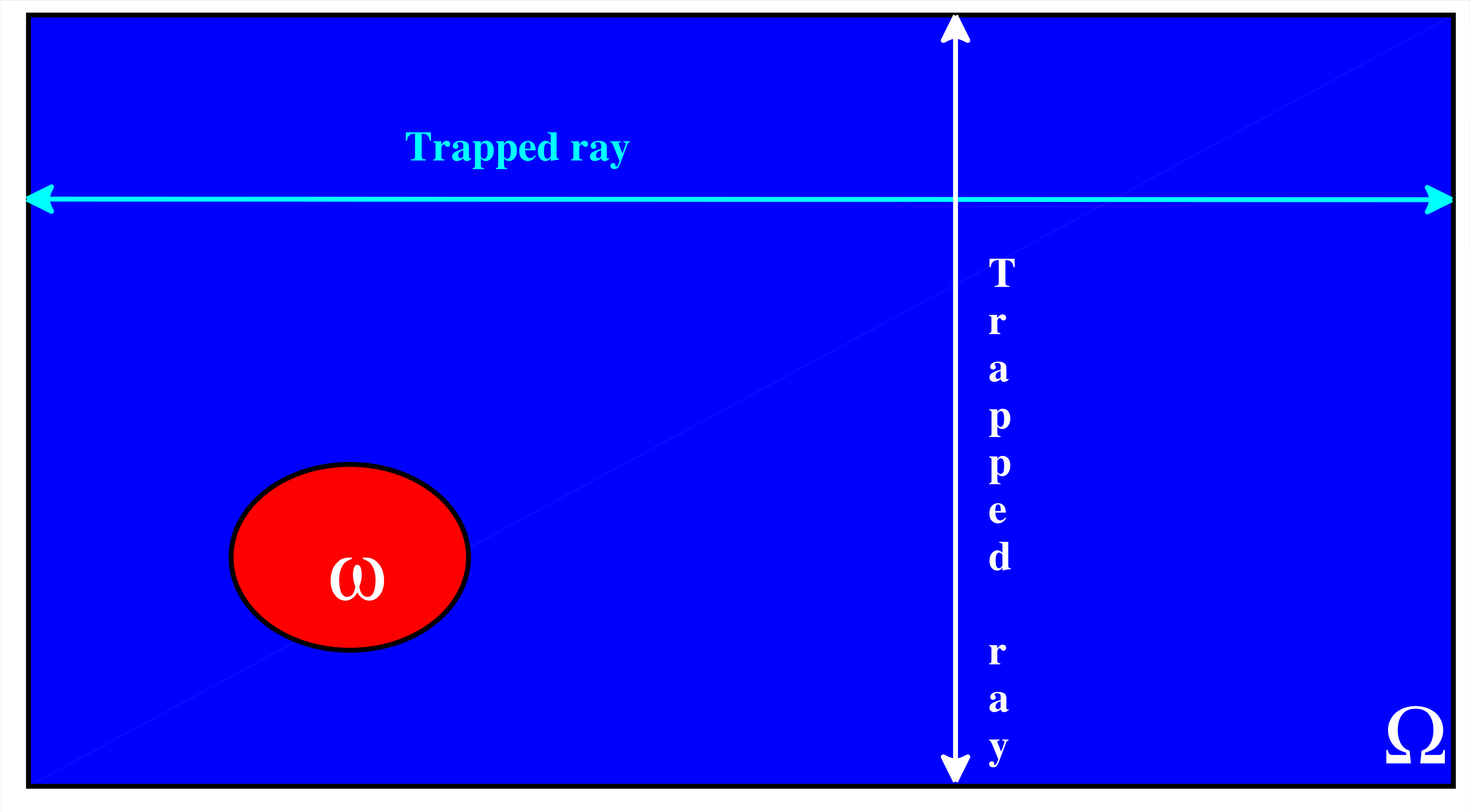

*Rays propagating inside the domain $\Omega$ following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools from Microlocal Analysis.*

# Qualitative change from $1-d$ to multi-$d$



A trapped ray scapping the observation region $\omega$ makes it impossible these observability inequalities to hold.



This ray analysis yields sharp results for wave propagation phenomena.

Its link to diffusion processes is less clear....

In a joint work with S. Ervedoza (ARMA, 2011) we have shown that, whenever the GCC is fulfilled for the wave equation, in time $T$, then we have the following upper bound for the diffusion process:

$$A \leq T^2/8.$$

Note that for a ball $\Omega$, with control on a neighborhood of the boundary,

$$T = 2\ell.$$

We thus get the sharp upper bound in this case:

$$A \leq \ell^2/2.$$

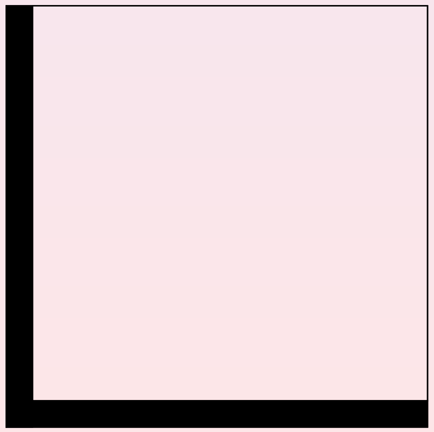
We use an **inverse Kannai transform.**

Our proof is based on an inverse Kannai transform that, to the best of our knowledge, was unknown until now:

$$W(s) = \int_{\mathbf{R}_+} \frac{1}{(4\pi t)^{1/2}} \sin\left(\frac{sS}{2t}\right) \exp\left(\frac{s^2 - S^2}{4t}\right) U(t)\, dt, \quad -S < s < S.$$

Note however, that, even under the GCC, except for the case of the radially symmetric geometry, there are no sharp upper bounds for other domains. For instance for **the square with observation on two consecutive sides** we have:

$$\frac{1}{2} \leq A \leq 1.$$



The kernel employed in this Kannai transform is characterized by the fact that

$$\begin{cases} \partial_t k(t,s) + \partial_{ss} k(t,s) = 0, & t \in \mathbf{R}_+,\ s \in (-S,S), \\ k(0,s) = 0, & s \in (-S,S), \\ \lim_{t\to\infty} |k(t,s)| = 0, & s \in (-S,S). \end{cases} \tag{5}$$

A particular solution is given by:

$$k(t,s) = \frac{1}{(4\pi t)^{1/2}} \sin\left(\frac{sS}{2t}\right) \exp\left(\frac{s^2 - S^2}{4t}\right)$$

$$= \frac{1}{(4\pi t)^{1/2}} \sin\left(\frac{sS}{2t}\right) \exp\left(\frac{-S^2}{4t}\right) \exp\left(\frac{s^2}{4t}\right)$$

which can be viewed as an infinite order derivate of the Gaussian heat kernel.

Very much in the spirit of the Tychonoff singular solution of the heat equation.

It can also be obtained through the Appell transform: *If $v(x,t)$ solves the heat equation then so does*

$$w(x,t) = G(x,t)v(\frac{x}{t}, -\frac{1}{t})$$

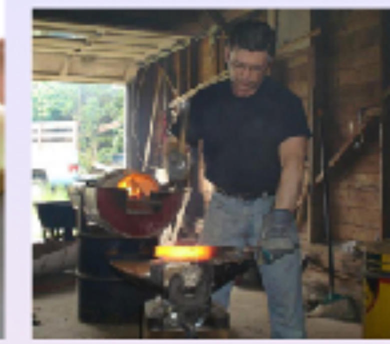
out of the particular solution

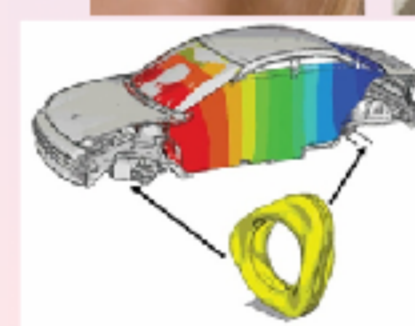
$$\exp(-\frac{S^2}{4}t)\exp(i\frac{S}{2}x).$$

## Why viscoelastic materials?

Viscoelastic materials are those for which the behavior combines liquid-like and solid-like characteristics. [1]

Viscoelasticity is important in areas such as **biomechanics**, **power industry** or **heavy construction**


- Synthetic polymers;
- Wood;
- Human tissue, cartilage;
- Metals at high temperature;
- Concrete, bitumen;
- ...

[1] See H. T. Banks, S. Hu and Z. R. Kenz, A Brief Review of Elasticity and Viscoelasticity for Solids, Adv. Appl. Math. Mech., Vol. 3, No. 1, 1-51.

# Viscoelasticity

A wave equation with both viscous Kelvin-Voigt and frictional damping:

$$\begin{aligned} y_{tt} - \Delta y - \Delta y_t + b(x) y_t &= 1_\omega h, \qquad x \in \Omega,\ t \in (0,T), && (6)\\ y &= 0, \qquad x \in \partial\Omega,\ t \in (0,T), && (7)\\ y(x,0) = y_0(x),\ y_t(x,0) &= y_1(x) \qquad x \in \Omega. && (8) \end{aligned}$$

Here, $\Omega$ is a smooth, bounded open set in $\mathbb{R}^N$, $b \in L^\infty(\Omega)$ is a given function determining the frictional damping and $h = h(x,t)$ is a control located in a open subset $\omega$ of $\Omega$.

We want to study the following problem:
Given $(y_0, y_1)$. Find a control $h$ such that the associated solution to (6)-(8) satisfies

$$y(T) = y_t(T) = 0.$$

# Viscoelasticity = Waves + Heat

$$y_{tt} - \Delta y - \Delta y_t = 0$$

$$=$$

$$y_{tt} - \Delta y = 0$$

$$+$$

$$\partial_t [y_t] - \Delta y_t = 0$$

Both equations are controllable. Should then the superposition be controllable as well?

Interesting open question: The role of splitting and alternating directions in the controllability of PDE.

## A geometric obstruction

Standard results on unique continuation do not apply.
The principal part of the operator is

$$\partial_t \Delta.$$

Then characteristic hyperplanes are of the form

$$t = t_0$$

and

$$x \cdot e = 1.$$

Vertical hyperplanes make it impossible to prove unique continuation from $\omega \times (0, T)$ towards the whole domain $\Omega$, even in the context of constant coefficients. Holmgren's uniqueness Theorem cannot be applied.
This phenomenon was previously observed by S. Micu in the context of the Benjamin-Bona-Mahoni equation [2] In that context the underlying operator is

$$\partial_t - \partial^3_{xxt}$$

but its principal part is the same

## Viscoelasticity = Heat + ODE

Note that

$$y_{tt} - \Delta y - \Delta y_t + y_t = (\partial_t - \Delta)(\partial_t + I).$$

$$\begin{aligned} y_t - \Delta y + (b(x) - 1)y &= z, && (9) \\ z_t + z &= 1_\omega h + (b(x) - 1)y, && (10) \\ y(x,t) = v(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T), && (11) \\ z(x,0) &= z_0(x), \quad x \in \Omega, && (12) \\ y(x,0) &= y_0(x), \quad x \in \Omega. && (13) \end{aligned}$$

In this form the controllability of the system is less clear. We are acting on the ODE variable $z$. But the control action does not allow to control the whole $z$. We are effectively acting on $y$ through $z$. What is the overall impact of the control?

## Viscoelasticity = Heat + ODE. Second version

Then

$$
\begin{aligned}
y_t + y &= v, && (14)\\
v_t - \Delta v &= 1_\omega h + (1 - b(x))(v - y), && (15)\\
v(x,t) = y(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times (0,T), && (16)\\
v(x,0) &= y_1(x) + y_0(x), \qquad x \in \Omega, && (17)\\
y(x,0) &= y_0(x), \qquad x \in \Omega. && (18)
\end{aligned}
$$

The question now becomes:
Given $(y_0, z_0)$. Find a control $h$ such that the associated solution to (14)-(18) satisfies

$$y(T) = v(T) = 0.$$

# Viscoelasticity = Heat + Memory

Note that

$$y_{tt} - \Delta y - \Delta y_t = \partial_t [y_t - \Delta y - \Delta \int_0^t y].$$

The later, heat with memory, was addressed by Gurrero and Imanuvilov[3], showing that the system is not null controllable.

[3] S. Guerrero, O. Yu. Imanuvilov, Remarks on non controllability of the heat equation with memory, ESAIM: COCV, 19 (1)(2013), 288–300.

## The case $b \equiv 1$

When $b \equiv 1$, the system reads:

$$\begin{aligned} v_t - \Delta v &= 1_\omega h, \\ y_t + y &= v. \end{aligned} \tag{19}$$

But we can consider the system with an added ficticious control:

$$\begin{aligned} v_t - \Delta v &= 1_\omega h, \\ y_t + y &= v + 1_\omega k. \end{aligned} \tag{20}$$

Control in two steps:

- Use the control $h$ to control $v$ to zero in time $T/2$.
- Then use the control $k$ to control the ODE dynamics in the time-interval $[T/2, T]$.

Warning. The second step cannot be fulfilled since the ODE does not propagate the action of the controller which is confined in $\omega$.
Possible solution: Make the control in the second equation move or, equivalently, replace the ODE by a transport equation.

This strategy was introduced and found to be successful in

P. Martin, L. Rosier, P. Rouchon, Null Controllability of the Structurally Damped Wave Equation with Moving Control, SIAM J. Control Optim., 51 (1)(2013), 660–684.
L. Rosier, B.-Y. Zhang, Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain, J. Differential Equations 254 (2013), 141-178.

by using Fourier series decomposition.

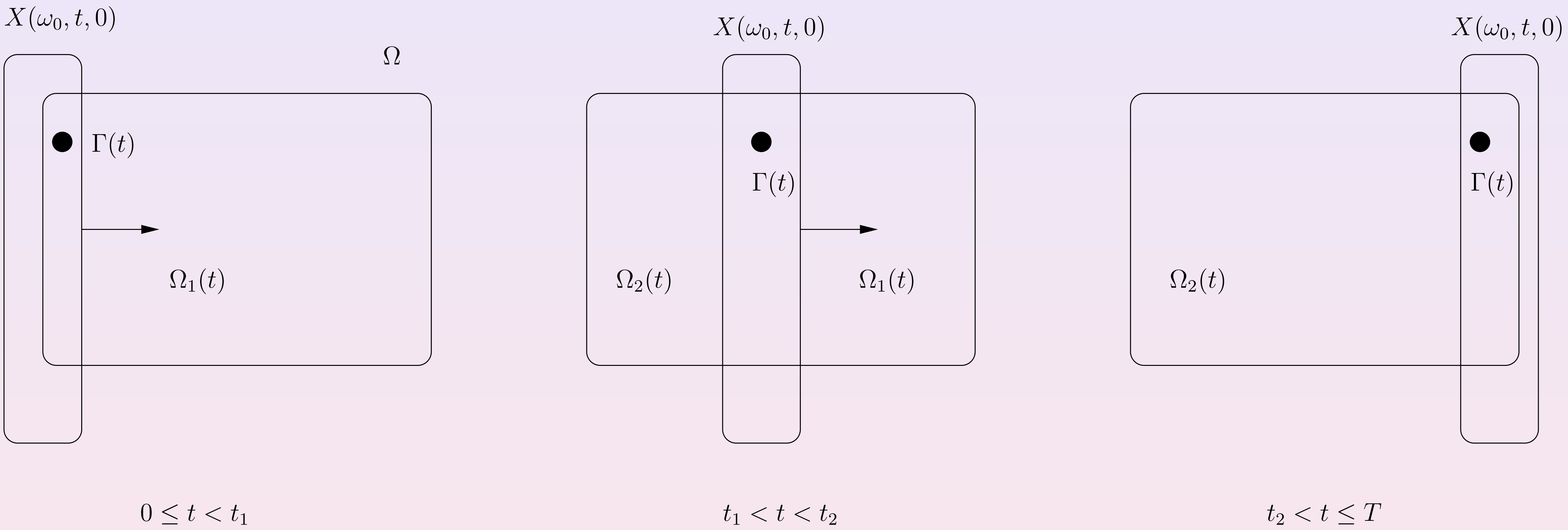
In the context of the example under consideration, if we make the control set $\omega$ move to $\omega(t)$ with a velocity field $a(t)$, then the ODE becomes:

$$y_t + a(t) \cdot \nabla y = 1_\omega k.$$

And it is sufficient that all characteristic lines pass by $\omega$or, in other words, that the set $\omega(t)$ covers the whole domain $\Omega$ in its motion.

Question: How to prove this kind of result in a more general setting where $b \neq 1$ so that the system does not decouple?

# An example of moving support of the control



## Observability

We consider the dual problem of (21)-(25):

$$\begin{aligned}
-p_t - \Delta p + (b(x)-1)p &= (b(x)-1)q, \quad (x,t) \in \Omega \times (0,T), && (21)\\
-q_t + q &= p, \quad (x,t) \in \Omega \times (0,T), && (22)\\
p(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T), && (23)\\
p(x,T) &= p_0(x), \quad x \in \Omega, && (24)\\
q(x,T) &= q_0(x), \quad x \in \Omega. && (25)
\end{aligned}$$

The null controllability property i equivalent to the following observability one

$$||p(0)||^2 + ||q(0)||^2 \le C \int_0^T \int_\omega |q|^2 dxdt, \qquad (26)$$

for all solutions of (21)-(25).

But the structure of the underlying PDE operator and, in particular, the existence of time-like characteristic hyperplanes, makes impossible the propagation of information in the space-like directions, thus making the

## Lack of observability for $b \equiv 1$

$$-p_t - \Delta p = 0 \quad , \qquad (x,t) \in \Omega \times (0,T), \tag{27}$$

$$-q_t + q = p, \qquad (x,t) \in \Omega \times (0,T), \tag{28}$$

It is impossible that

$$||p(0)||^2 + ||q(0)||^2 \leq C \int_0^T \int_\omega |q|^2 dxdt, \tag{29}$$

## Remedy: Moving control

Let us assume that $\omega \equiv \omega(t)$.
The controllable system under consideration then reads:

$$
\begin{aligned}
y_t - \Delta y + (b(x) - 1)y &= z, && (30)\\
z_t + z &= 1_{\omega(t)} h + (b(x) - 1)y, && (31)\\
y(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T), && (32)\\
z(x,0) &= z_0(x), \quad x \in \Omega, && (33)\\
y(x,0) &= y_0(x), \quad x \in \Omega. && (34)
\end{aligned}
$$

## Motion of the support of the control

In practice, the trajectory of the control can be taken to be determined by the flow $X(x,t,t_0)$ generated by some vector field $f \in C([0,T]; W^{2,\infty}(\mathbb{R}^N;\mathbb{R}^N))$, i.e. $X$ solves

$$\begin{cases} \dfrac{\partial X}{\partial t}(x,t,t_0) = f(X(x,t,t_0),t), \\ X(x,t_0,t_0) = x. \end{cases} \tag{35}$$

**Admissible trajectories**: There exist a bounded, smooth, open set $\omega_0 \subset \mathbb{R}^N$, a curve $\Gamma \in C^\infty([0,T];\mathbb{R}^N)$, and two times $t_1, t_2$ with $0 \le t_1 < t_2 \le T$ such that:

$$\Gamma(t) \in X(\omega_0,t,0) \cap \Omega, \quad \forall t \in [0,T]; \tag{36}$$

$$\overline{\Omega} \subset \cup_{t\in[0,T]} X(\omega_0,t,0) = \{X(x,t,0); \ \ x \in \omega_0, \ t \in [0,T]\}; \tag{37}$$

$$\Omega \setminus \overline{X(\omega_0,t,0)} \text{ is nonempty and connected for } t \in [0,t_1] \cup [t_2,T]; \tag{38}$$

$$\Omega \setminus \overline{X(\omega_0,t,0)} \text{ has two connected components for } t \in (t_1,t_2); \tag{39}$$

# A failing moving support

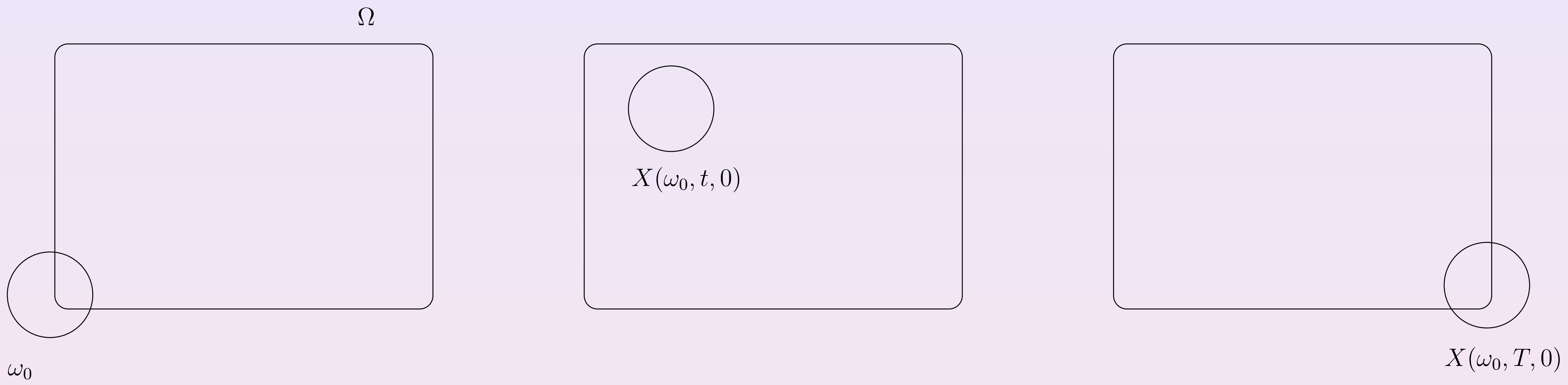
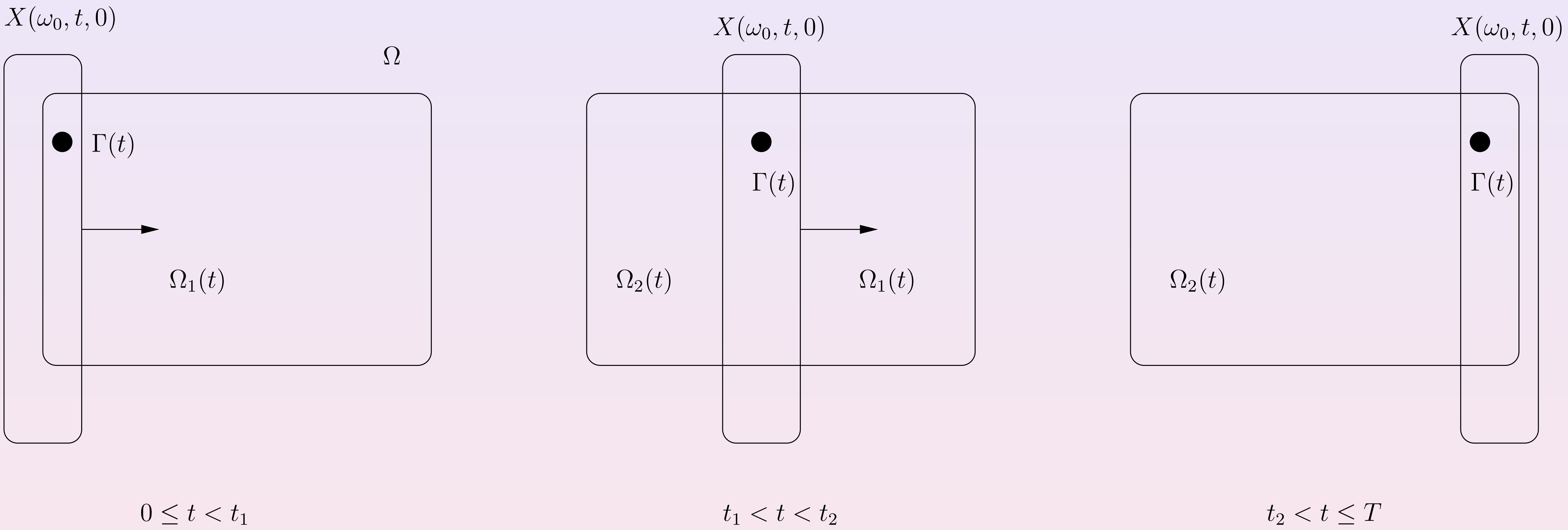



Figure: Example for which condition (39) fails.

**Remark**: Note that it would be OK for $b \equiv 1$.

# A successful motion



# Proof of the observability inequality

Our strategy is based on the use of Carleman inequalities for the heat and the ODE.

Two main difficulties appear:

1. Carleman inequalities for heat and ODE equations with a moving control region;
2. We must have the same weight functions in the Carleman for both equations.

Fortunately, we can handle both difficulties. Note that similar strategies were implemented successfully for the system of thermoelasticity in

P. Albano, D. Tataru, Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system, Electron. J. Differential Equations, 22 (2000), 1–15.

There exist some constants $\lambda_0 > 0$, $s_0 > 0$ and $C_0 > 0$ such that for all $\lambda \geq \lambda_0$, all $s \geq s_0$ and all $p \in C([0,T]; L^2(\Omega))$ with $p_t + \Delta p \in L^2(0,T; L^2(\Omega))$, the following holds

$$\int_0^T\int_\Omega [(s\theta)^{-1}(|\Delta p|^2 + |p_t|^2) + \lambda^2(s\theta)|\nabla p|^2 + \lambda^4(s\theta)^3|p|^2]e^{-2s\varphi}dxdt$$
$$\leq C_0\left(\int_0^T\int_\Omega |p_t + \Delta p|^2 e^{-2s\varphi}dxdt + \int_0^T\int_{\omega_1(t)} \lambda^4(s\theta)^3|p|^2e^{-2s\varphi}dxdt\right), \tag{41}$$

for all $\overline{\omega_0} \subset \omega_1$.
Similarly:

$$\int_0^T\int_\Omega (\lambda^2 s\theta)|q|^2 e^{-2s\varphi}dxdt$$
$$\leq C_1\Big(\int_0^T\int_\Omega |q_t|^2 e^{-2s\varphi}dxdt + \int_0^T\int_{\omega(t)} \lambda^2(s\theta)^2|q|^2e^{-2s\varphi}dxdt.$$

In the previous Lemmas, the weights have the form:

$$\varphi(x,t) = g(t)(e^{\frac{3}{2}\lambda||\psi||_{L^\infty}} - e^{\lambda\psi(x,t)}) \sim \frac{(e^{\frac{3}{2}\lambda||\psi||_{L^\infty}} - e^{\lambda\psi(x,t)})}{t(T-t)}, \quad (42)$$

$$\theta(x,t) = g(t)e^{\lambda\psi(x,t)} \sim \frac{e^{\lambda\psi(x,t)}}{t(T-t)}, \quad (43)$$

where $\psi \in C^\infty(\overline{\Omega} \times [0,T])$ is a weight having the following properties:

There exist a number $\delta \in (0, T/2)$ such that

$$\nabla\psi(x,t) \neq 0, \qquad t \in [0, T],\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \tag{44}$$

$$\psi_t(x,t) \neq 0, \qquad t \in [0, T],\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \tag{45}$$

$$\psi_t(x,t) > 0, \qquad t \in [0, \delta],\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \tag{46}$$

$$\psi_t(x,t) < 0, \qquad t \in [T-\delta, T],\ x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \tag{47}$$

$$\frac{\partial\psi}{\partial n}(x,t) \leq 0, \qquad t \in [0, T],\ x \in \partial\Omega, \tag{48}$$

$$\psi(x,t) > \frac{3}{4}||\psi||_{L^\infty(\Omega\times(0,T))}, \qquad t \in [0, T],\ x \in \overline{\Omega}. \tag{49}$$

for all

$$\overline{\omega_0} \subset \omega_1, \quad \overline{\omega_1} \subset \omega.$$

**Remark**: Basically, $\psi$ drags the critical points of $\psi(x,0)$ inside the control region during the evolution of the flow.

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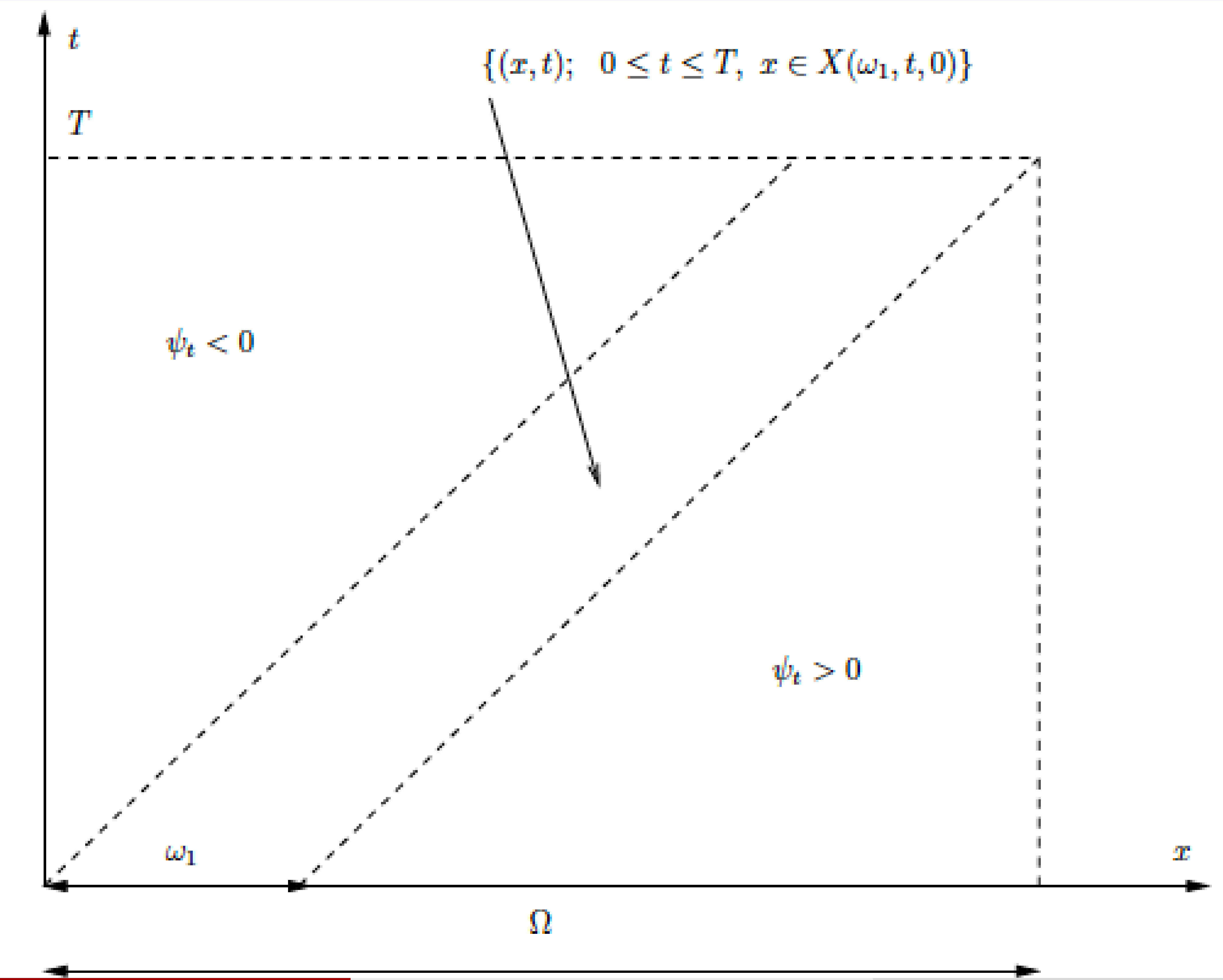
$$\overline{\omega_0} \subset \omega_1, \quad \overline{\omega_1} \subset \omega.$$

**Remark**: Basically, $\psi$ drags the critical points of $\psi(x,0)$ inside the control region during the evolution of the flow.

# Obstruction for the weight function

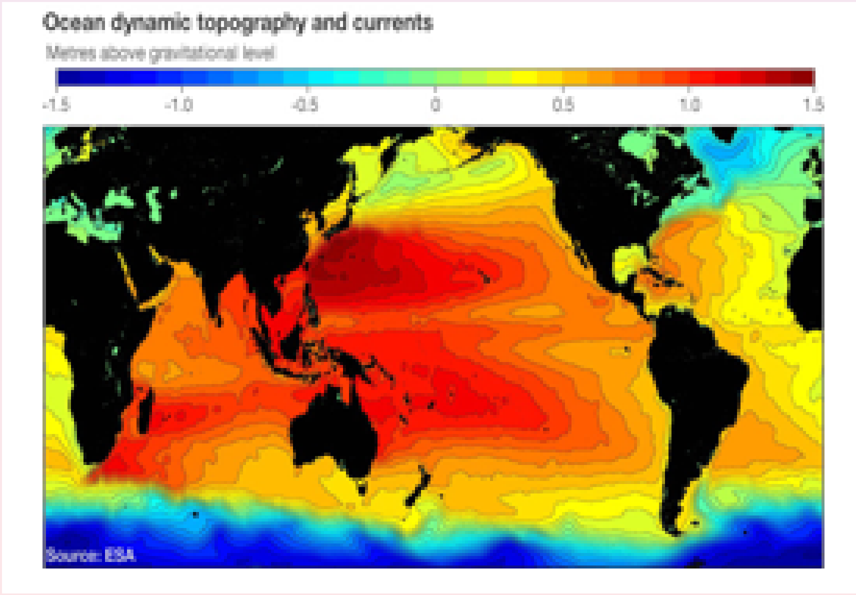


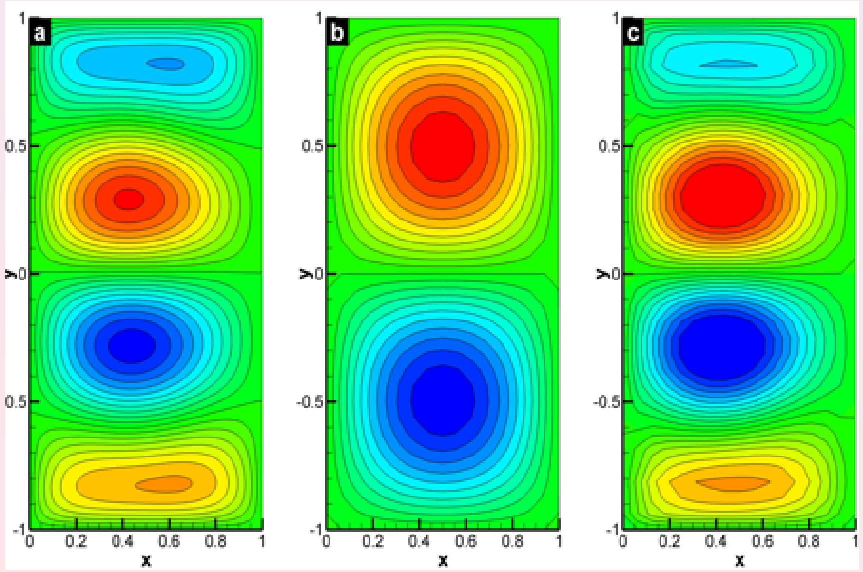
# Final comments

- Can the technical geometric assumptions on the moving control be removed?
- Can one derive similar results by simply assuming that the support of the control covers the whole domain?
- To which extent this methodology can be applied in problems where there are vertical characteristic hyperplanes (BBM, heat with memory,...)?
- Other models with memory.
- Nonlinear versions.

## Climate modelling

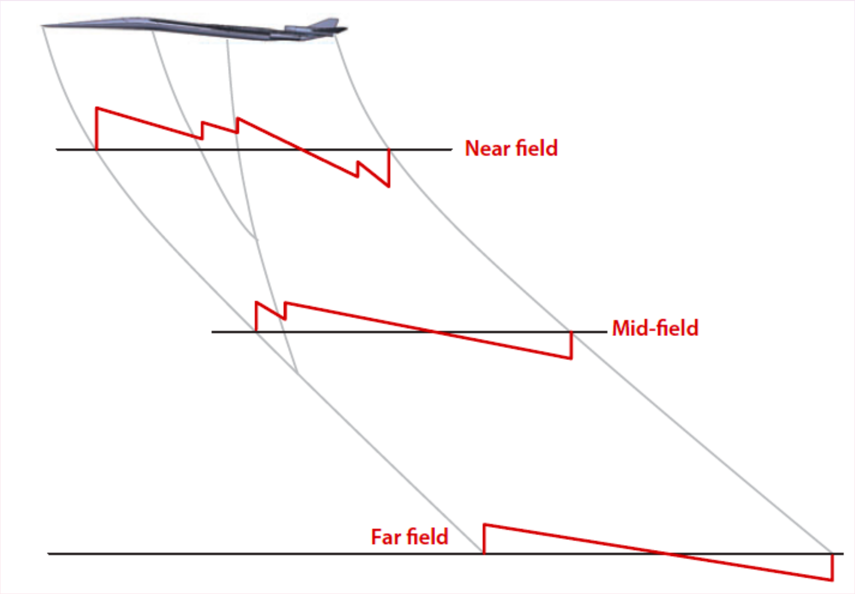
- Climate modeling is a grand challenge computational problem, a research topic at the frontier of computational science.
- Simplified models for geophysical flows have been developed aim to: capture the important geophysical structures, while keeping the computational cost at a minimum.
- Although successful in numerical weather prediction, these models have a prohibitively high computational cost in climate modeling.



# Sonic boom

- Goal: the development of supersonic aircraft that are sufficiently quiet so that they can be allowed to fly supersonically over land.
- The pressure signature created by the aircraft must be such that, when it reaches the ground, (a) it can barely be perceived by the human ear, and (b) it results in disturbances to man-made structures that do not exceed the threshold of annoyance for a significant percentage of the population.




Juan J. Alonso and Michael R. Colonno, Multidisciplinary Optimization with Applications to Sonic-Boom Minimization, Annu. Rev. Fluid Mech. 2012, 44:505 – 26.

# Geometric integration

## Long time numerics: Geometric/Symplectic integration
## Numerical integration of the pendulum (A. Marica)[a]

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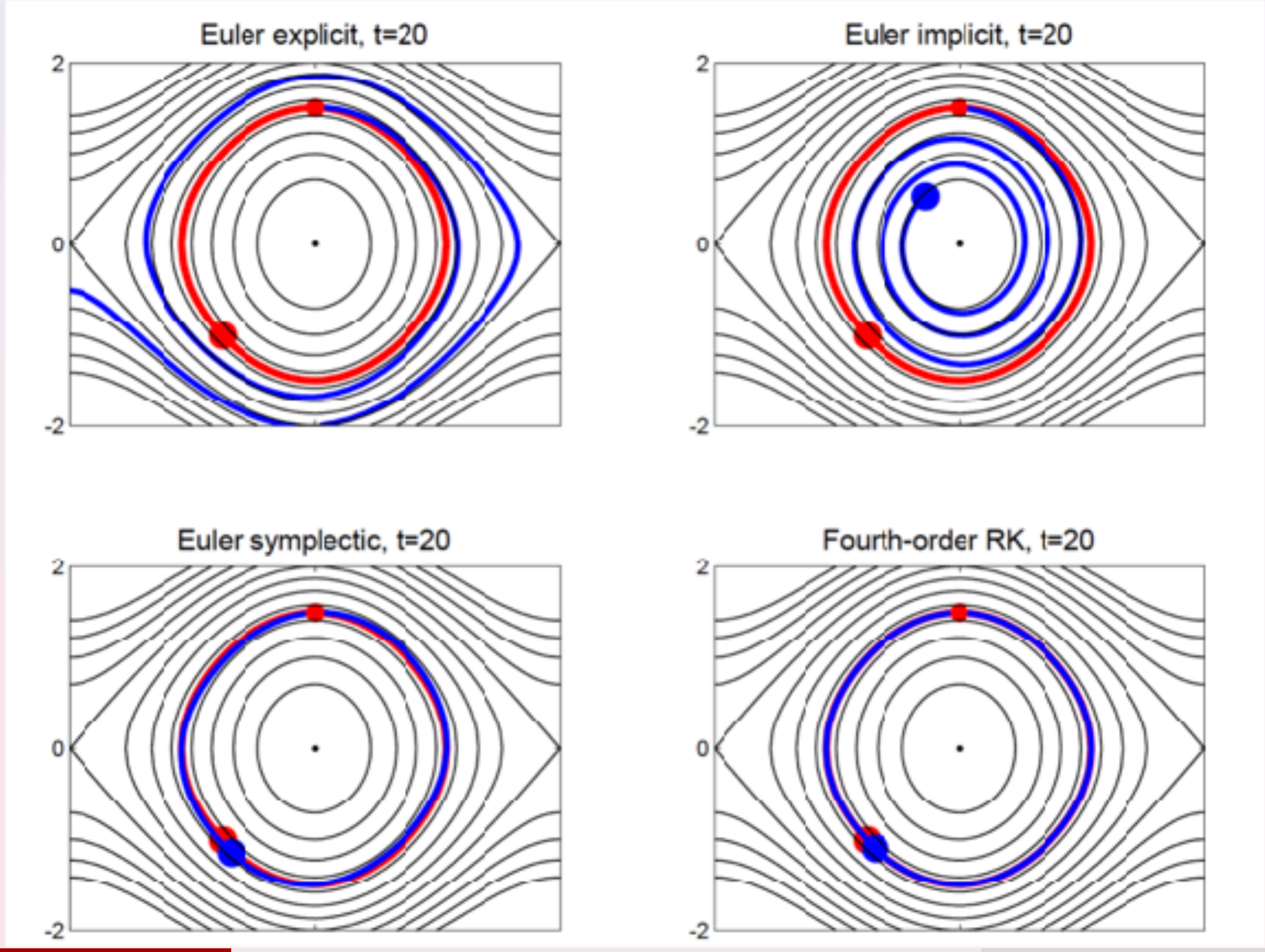
[a] HAIRER, E., LUBICH, Ch., WANNER, G.. Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations. 2nd ed. Berlin : Springer, 2006, 644 p.



## Joint work with L. Ignat & A. Pozo

Consider the 1-D conservation law with or without viscosity:

$$u_t + \left[u^2\right]_x = \varepsilon u_{xx}, x \in \mathbb{R}, t > 0.$$

Then[4]:

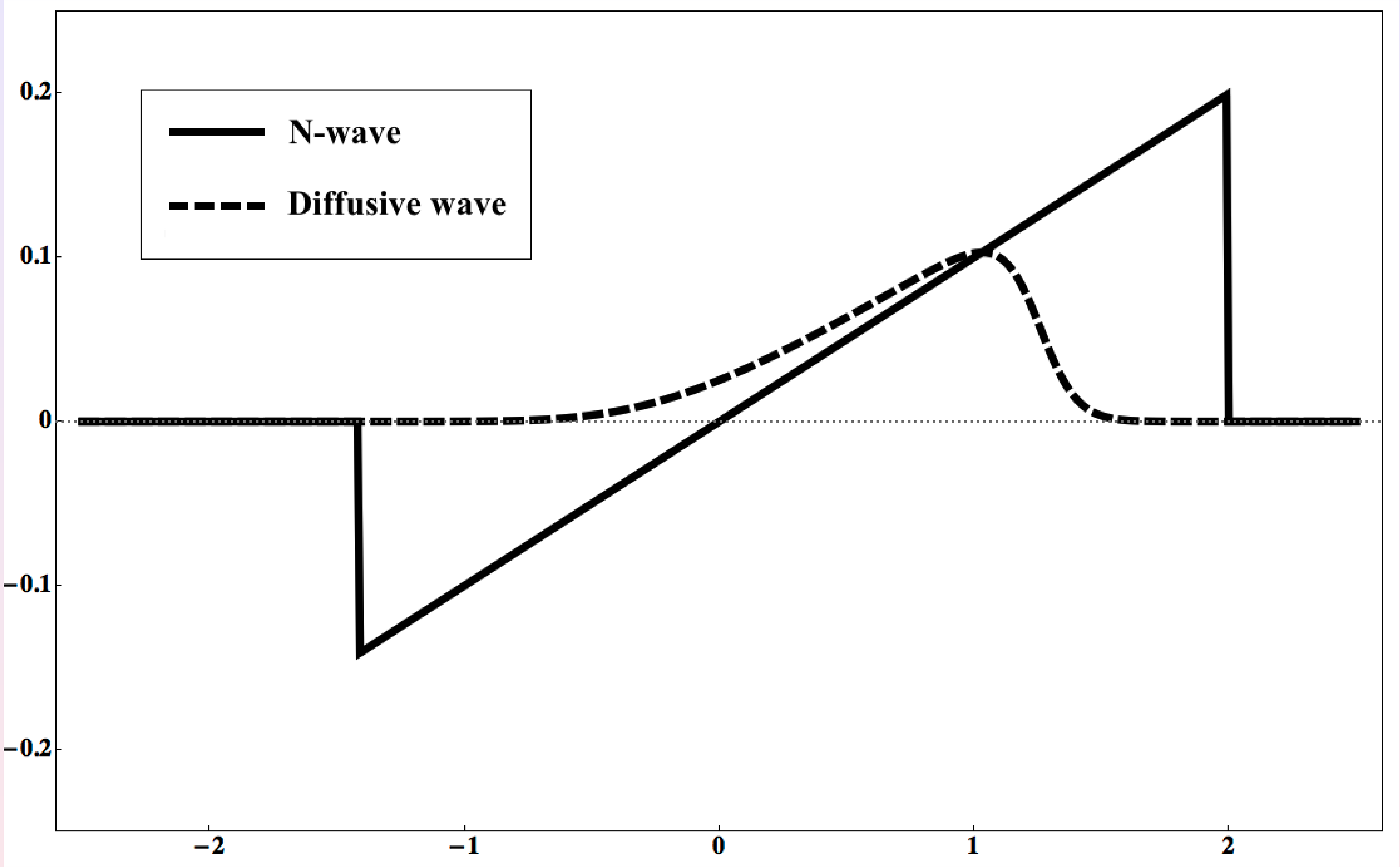
- If $\varepsilon = 0$, $u(\cdot, t) \sim N(\cdot, t)$ as $t \to \infty$;
- If $\varepsilon > 0$, $u(\cdot, t) \sim u_M(\cdot, t)$ as $t \to \infty$,

$u_M$ is the constant sign self-similar solution of the viscous Burgers equation (defined by the mass $M$ of $u_0$), while $N$ is the so-called hyperbolic N-wave.

In both cases:

$$u(x, t) \sim t^{-1/2} F(x/\sqrt{t}),\ t \to \infty.$$

[4] Y. K. Kim and A. Tzavaras, Diffusive $N$-waves and metastability in the Burgers equation, SIAM J. Math. Anal., 33 (3), 607 – 633.


N-wave
Diffusive wave
0.2
0.1
0
−0.1
−0.2
−2
−1
0
1
2

# Conservative schemes for the inviscid equation

Let us consider now numerical approximation schemes

$$\begin{cases} u_j^{n+1} = u_n^j - \dfrac{\Delta t}{\Delta x}\left(g_{j+1/2}^n - g_{j-1/2}^n\right), & j \in \mathbf{Z}, \mathbf{n} > \mathbf{0}. \\ u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, & j \in \mathbf{Z}, \end{cases}$$

The approximated solution $u_\Delta$ is given by

$$u_\Delta(t,x) = u_j^n, \quad x_{j-1/2} < x < x_{j+1/2},\ t_n \leq t < t_{n+1},$$

where $t_n = n\Delta t$ and $x_{j+1/2} = (j + \frac{1}{2})\Delta x$.

Is the large tine dynamics of these discrete systems, a discrete version of the continuous one?

## 3-point conservative schemes

1. Lax-Friedrichs

$$g^{LF}(u,v) = \frac{u^2+v^2}{4} - \frac{\Delta x}{\Delta t}\left(\frac{v-u}{2}\right),$$

2. Engquist-Osher

$$g^{EO}(u,v) = \frac{u(u+|u|)}{4} + \frac{v(v-|v|)}{4},$$

3. Godunov

$$g^{G}(u,v) = \begin{cases} \min\limits_{w\in[u,v]} \frac{w^2}{2}, & \text{if } u \le v, \\ \max\limits_{w\in[v,u]} \frac{w^2}{2}, & \text{if } v \le u. \end{cases}$$

## Numerical viscosity

We can rewrite three-point monotone schemes in the form

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{(u_{j+1}^n)^2-(u_{j-1}^n)^2}{4\Delta x}=R(u_j^n,u_{j+1}^n)-R(u_{j-1}^n,u_j^n)$$

where the numerical viscosity $R$ can be defined in a unique manner as

$$R(u,v)=\frac{Q(u,v)(v-u)}{2}=\frac{\lambda}{2}\Big(\frac{u^2}{2}+\frac{v^2}{2}-2g(u,v)\Big).$$

For instance:

$$R^{LF}(u,v)=\frac{v-u}{2},$$

$$R^{EO}(u,v)=\frac{\lambda}{4}(v|v|-u|u|),$$

$$R^{G}(u,v)=\begin{cases}\frac{\lambda}{4}\mathrm{sign}(|u|-|v|)(v^2-u^2), & v\le 0\le u,\\ \frac{\lambda}{4}(v|v|-u|u|). & \text{elsewhere.}\end{cases}$$

# Properties

These three schemes are well-known to satisfy the following properties:

- They converge to the entropy solution
- They are monotonic
- They preserve the total mass of solutions
- They are OSLC consistent:

$$\frac{u_{j-1}^n - u_{j+1}^n}{2\Delta x} \leq \frac{2}{n\Delta t}$$

- $L^1 \to L^\infty$ decay with a rate $O(t^{-1/2})$
- Similarly they verify uniform $BV_{\text{loc}}$ estimates

**But do they capture correctly the asymptotic behavior of solutions as $t \to \infty$?**

# Main result: Viscous effective behavior

**Theorem (Lax-Friedrichs scheme)**

Consider $u_0 \in L^1(\mathbf{R})$ and $\Delta x$ and $\Delta t$ such that $\lambda \left| u^n \right|_{\infty,\Delta} \leq 1$, $\lambda = \Delta t/\Delta x$. Then, for any $p \in [1, \infty)$, the numerical solution $u_\Delta$ given by the Lax-Friedrichs scheme satisfies

$$\lim_{t \to \infty} t^{\frac{1}{2}(1-\frac{1}{p})} \left| u_\Delta(t) - w(t) \right|_{L^p(\mathbb{R})} = 0,$$

where the profile $w = w_{M_\Delta}$ is the unique solution of

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = \frac{(\Delta x)^2}{2} w_{xx}, & x \in \mathbf{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u^0_\Delta$.

## Why?

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{(u_{j+1}^n)^2-(u_{j-1}^n)^2}{4\Delta x}=R^{LF}(u_j^n,u_{j+1}^n)-R^{LF}(u_{j-1}^n,u_j^n)$$

with

$$R^{LF}(u,v)=\frac{v-u}{2}.$$

Thus

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{(u_{j+1}^n)^2-(u_{j-1}^n)^2}{4\Delta x}\sim\frac{1}{2}\left[u_{j+1}^n+u_{j-1}^n-2u_j^n\right]\sim\frac{(\Delta x)^2}{2}u_{xx}.$$

# Main result: Inviscid effective behavior

## Theorem (Engquist-Osher and Godunov schemes)

Consider $u_0 \in L^1(\mathbf{R})$ and $\Delta x$ and $\Delta t$ such that $\lambda \Big| u^n \Big|_{\infty,\Delta} \leq 1$, $\lambda = \Delta t / \Delta x$. Then, for any $p \in [1, \infty)$, the numerical solutions $u_\Delta$ given by Engquist-Osher and Godunov schemes satisfy the same asymptotic behavior but for the hyperbolic $N - wave$ $w = w_{p_\Delta, q_\Delta}$ unique solution of

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = 0, & x \in \mathbf{R}, t > 0, \\ w(0) = M_\Delta \delta_0, & \lim_{t \to 0} \int_0^x w(t,z)dz = \begin{cases} 0, & x < 0, \\ -p_\Delta, & x = 0, \\ q_\Delta - p_\Delta, & x > 0, \end{cases} \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$ and
$p_\Delta = -\min_{x \in \mathbb{R}} \int_{-\infty}^x u_\Delta^0(z)dz$ and $q_\Delta = \max_{x \in \mathbb{R}} \int_x^\infty u_\Delta^0(z)dz$.

# Proof

Scaling transformation:

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

The asymptotic behavior of $u(x,t)$ as $t \to \infty$ is reduced to the analysis of the behavior of the rescaled family $u_\lambda$ as $\lambda \to \infty$ but in the finite time horizon $0 < t < 1$.

# Example

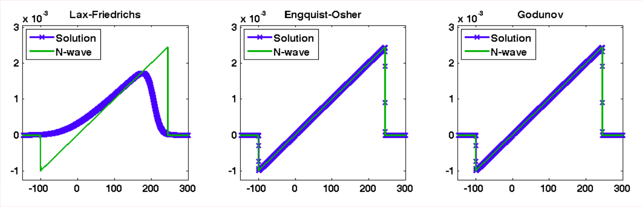
Let us consider the inviscid Burgers equation with initial data

$$u_0(x) = \begin{cases} -0.05, & x \in [-1, 0], \\ 0.15, & x \in [0, 2], \\ 0, & \text{elsewhere.} \end{cases}$$

The parameters that describe the asymptotic N-wave profile are:

$$M = 0.25 \ , \quad p = 0.05 \quad \text{and} \quad q = 0.3.$$

We take $\Delta x = 0.1$ as the mesh size for the interval $[-350, 800]$ and $\Delta t = 0.5$. Solution to the Burgers equation at $t = 10^5$:



## Similarity variables

Let us consider the change of variables given by:

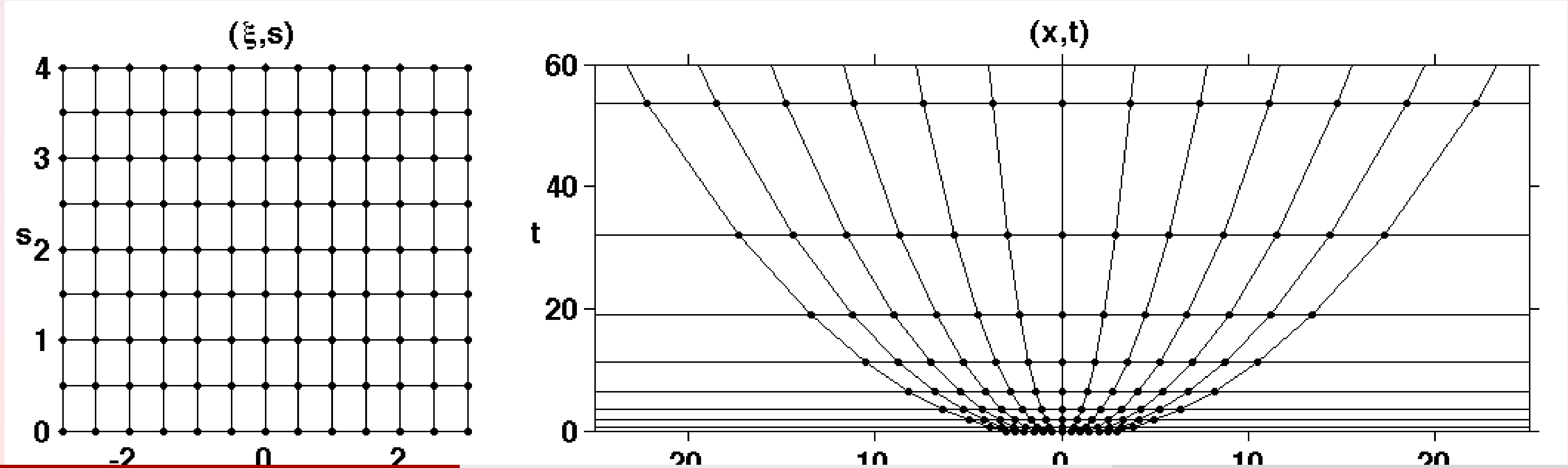
$$s = \ln(t+1), \qquad \xi = x/\sqrt{t+1}, \qquad w(\xi, s) = \sqrt{t+1}\ u(x,t),$$

which turns the continuous Burgers equation into

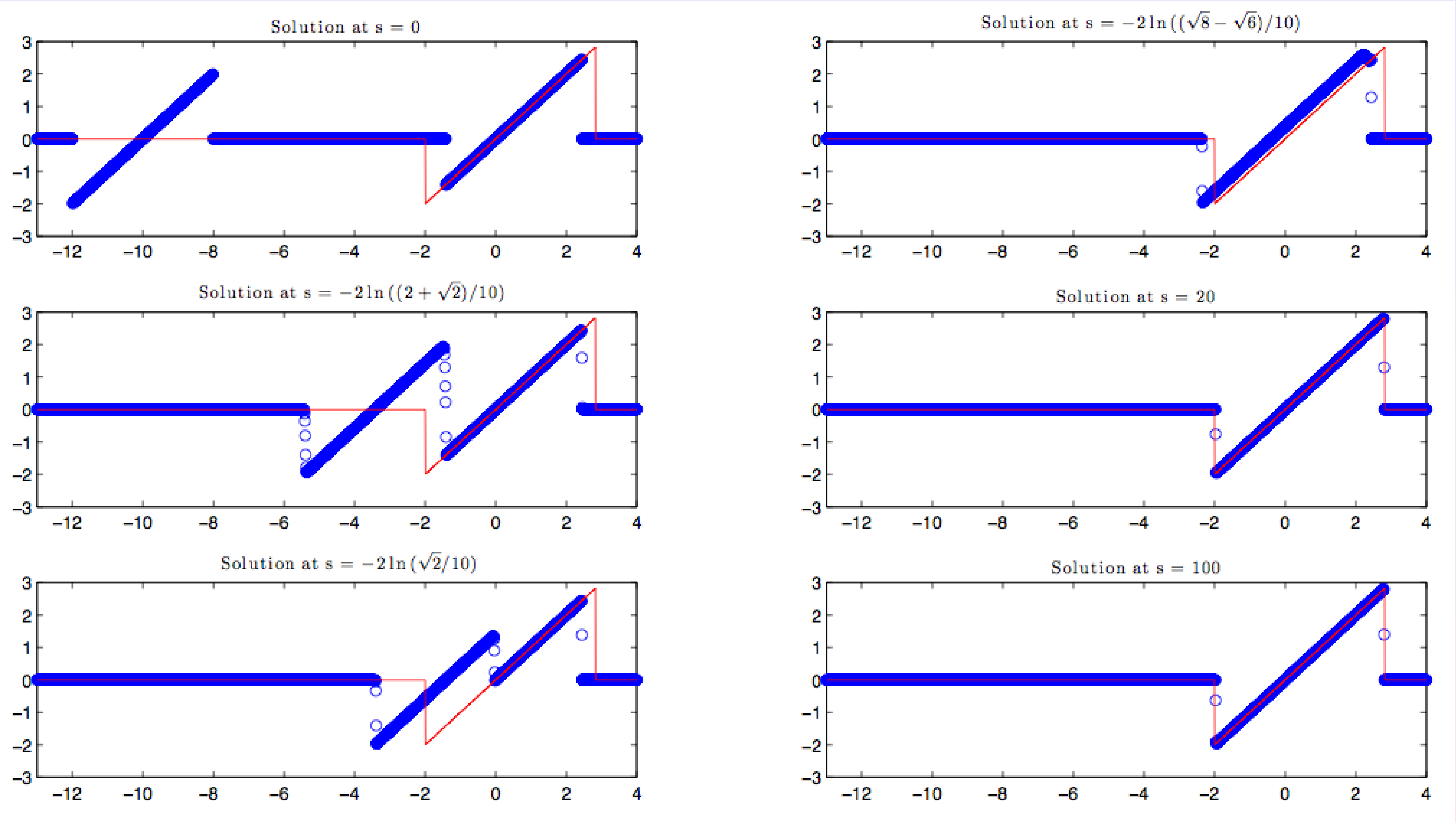
$$w_s + \left(\frac{1}{2}w^2 - \frac{1}{2}\xi w\right)_\xi = 0, \quad \xi \in \mathbf{R}, s > 0.$$

The asymptotic profile of the N-wave becomes a steady-state solution:

$$N_{p,q}(\xi) = \begin{cases} \xi, & -\sqrt{2p} < \xi < \sqrt{2q}, \\ 0, & \text{elsewhere}, \end{cases}$$



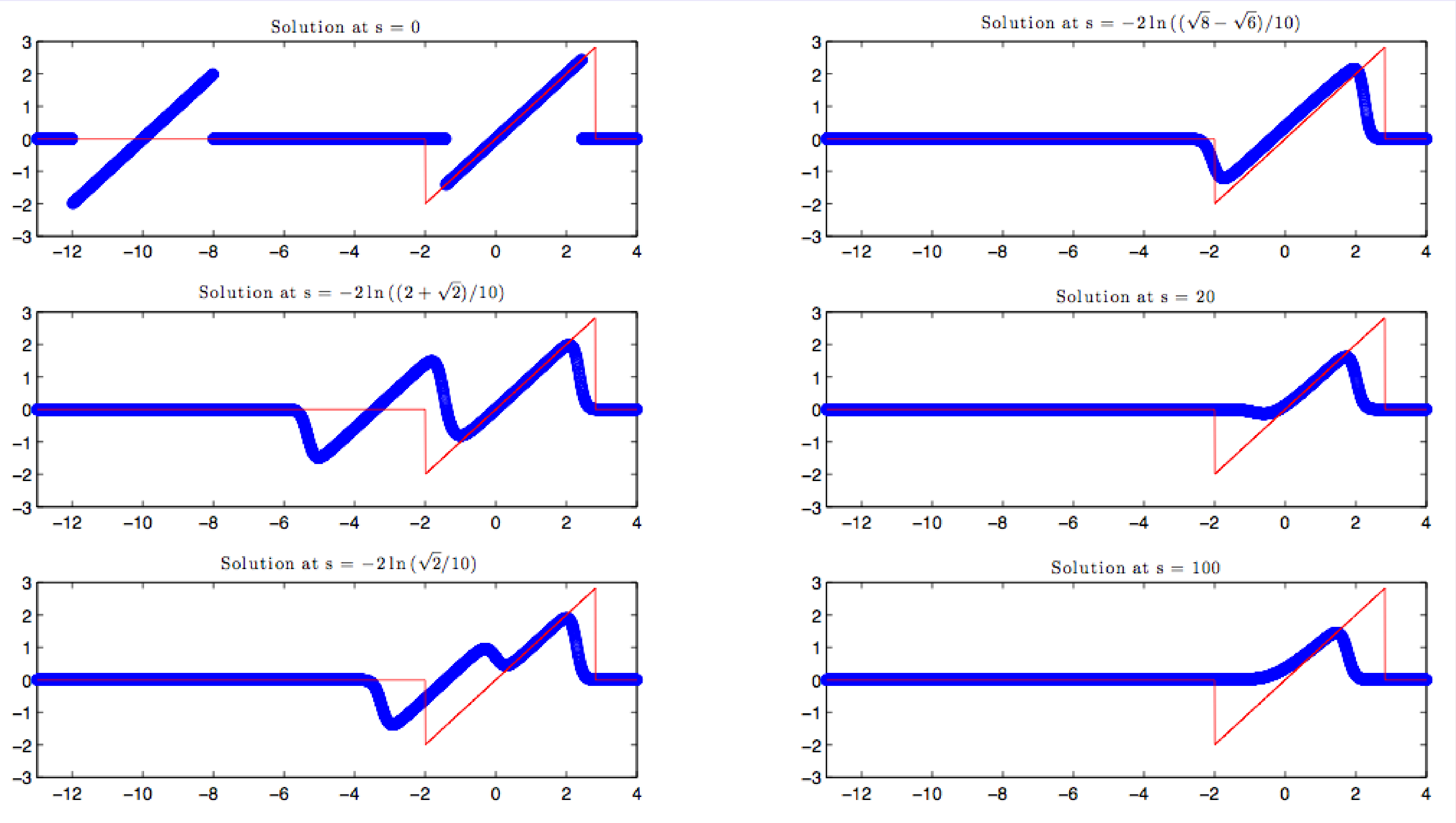
## Examples




Convergence of the numerical solution using Engquist-Osher scheme (circle dots) to the asymptotic N-wave (solid line). We take $\Delta\xi = 0.01$ and $\Delta s = 0.0005$.

Snapshots at $s = 0$, $s = 2.15$, $s = 3.91$, $s = 6.55$, $s = 20$ and $s = 100$.

# Examples




Numerical solution using the Lax-Friedrichs scheme (circle dots), taking $\Delta\xi = 0.01$ and $\Delta s = 0.0005$. The N-wave (solid line) is not reached, as it converges to the diffusion wave.

Snapshots at $s = 0$, $s = 2.15$, $s = 3.91$, $s = 6.55$, $s = 20$ and $s = 100$.

## Physical vs. Similarity variables

Comparison of numerical and exact solutions at $t = 1000$. We choose $\Delta\xi$ such that the $\big|\cdot\big|_{1,\Delta}$ error is similar. The time-steps are $\Delta t = \Delta x/2$ and $\Delta s = \Delta\xi/20$, respectively, enough to satisfy the CFL condition.

For $\Delta x = 0.1$:

| | Nodes | Time-steps | $\big\|\cdot\big\|_{1,\Delta}$ | $\big\|\cdot\big\|_{2,\Delta}$ | $\big\|\cdot\big\|_{\infty,\Delta}$ |
|---|---|---|---|---|---|
| Physical | 1501 | 19987 | 0.0867 | 0.0482 | 0.0893 |
| Similarity | 215 | 4225 | 0.0897 | 0.0332 | 0.0367 |

For $\Delta x = 0.01$:

| | Nodes | Time-steps | $\big\|\cdot\big\|_{1,\Delta}$ | $\big\|\cdot\big\|_{2,\Delta}$ | $\big\|\cdot\big\|_{\infty,\Delta}$ |
|---|---|---|---|---|---|
| Physical | 15001 | 199867 | 0.0093 | 0.0118 | 0.0816 |
| Similarity | 2000 | 39459 | 0.0094 | 0.0106 | 0.0233 |

# Conclusions

- Within the class of convergent numerical schemes we have shown the need of discriminating those that are asymptotically correct.
- We have shown the significant reduction on the computational cost when using the intrinsic similarity variables.