Lecture 1
Trace Inequality for Fractional Integrals, and Related Topics

ALEXANDER MESKHI
Kutaisi International University
TSU A. Razmadze Mathematical Institute

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We discuss:

1. Trace inequality for fractional integrals (Riesz potentials, Riemann-Liouville operators);
2. Boundedness/compactness criteria for fractional integrals defined with respect to a general measure. Stein-Weiss-type Inequalities.
3. Sharp weighted norm estimates for Riesz potentials (sharp Olsen's inequality).

We deal with linear and multilinear fractional integrals.

Part 1
Trace Inequality for Riesz Potentials

Riesz Potentials

Riesz potentials:

$$I_\alpha f(x) = \gamma(n,\alpha) \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \quad x \in \mathbb{R}^n, \tag{0.1}$$

play important role in harmonic analysis and PDEs, for example, in theory of Sobolev Embeddings (see, e.g., monographs by [Mazya], [Adams and Hedberg], etc.)

Sobolev's Classical Inequality

Theorem

Let $1 < p, q < \infty$ *and* $0 < \alpha < n/p$. *Then* I_α *is bounded from* L^p *to* L^q *if and only if*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Sobolev exponent:

$$q = \frac{pn}{n - \alpha p}$$

Riesz Potentials

The appropriate fractional maximal operator is given by the formula:

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)| dy, \quad 0 < \alpha < n, \quad x \in \mathbb{R}^n.$$

Taking formally $\alpha = 0$, we have the Hardy–Littlewood maximal function $M_0 f = Mf$ which is significant in Harmonic Analysis, particularly in the theory of Singular integrals.

The following pointwise estimate is obvious:

$$M_\alpha f(x) \leq C_{\alpha,n} I_\alpha f(x), \quad f \geq 0,$$

however the inverse inequality holds in terms of norms.

Trace Inequality

Trace inequalities for Riesz potentials I_α deals with a Borel measure ν on $\mathbb{R}^n$ for which the inequality

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q d\nu(x)\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p} \tag{0.2}$$

Holds.

If ν is absolutely continuous, $d\nu(x) = v(x)dx$, where v is a non-negative locally integralble (weight), then the trace inequality is:

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q v(x) dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p} \tag{0.3}$$

Some applications:

1. Sobolev embeddings (Maz'ya, Adams and Hedberg).
2. Very profound impact of trace inequalities on spectral problems of differential operators, and in particular on eigenvalue estimates for Schrodinger operators (see e.g., M. Frazier, B. Jawerth and G. Weiss, Littlewood-Paley theory and the study of function spaces. Regional Conference Series in Mathematics, Vo79, Amer. Math. Soc. Providence; RI, 1991; R. Kerman and E. Sawyer, 1986.

1. Close connection with the solubility of certain semilinear differential operators with minimal restrictions on the regularity of the coefficients and data. In fact, the existence of positive solutions of certain nonlinear differential equations is equivalent to the validity of a certain two-weighted inequality for a potential-type operator, in which the weights are expressed in terms of coefficients and data.

D.R. Adams and M. Pierre, Capacitary strong type estimates in semilinear problemns. Ann . Inst. Fourier (Grenoble) 41(1991), 117-135.

P. Baras and M. Pierre, Critere d'existence de solutions positives pour des equations semi-lineaires non monotones. Ann. Inst. H. Poincare Anal. Non Lineaire 2(1985), 185- 212

K. Hansson, Imbedding theorems of Sobolev type in potential theory. Math. Scand. 45(1979), 77-102.

V. G. Maz'ya, I. E. Verbitsky, Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. Ark. Mat.33(1995), 81-115.

In examples 2 and 3 the diagonal case $p = q$ is essential.

In 1971 D. Adams proved that necessary and sufficient condition on ν guaranteeing (0.3) for $1 < p < q < \infty$ and $0 < \alpha < n/p$ is that measure ν satisfies the condition: there is a positive constant C such that for all balls $B \subset \mathbb{R}^n$,

$$\nu(B) \leq C|B|^{(1/p-\alpha/n)q}.$$

He also noticed that this condition is necessary but not sufficient for (0.3) in the diagonal case $p = q$ (see also Lemarie-Rieusset, 2012).

In the sequel we will use the notation:

$$[\nu]_{p,q,\alpha} := \sup_B \Big(\nu(B)\Big)^{1/q} |B|^{\alpha/n-1/p}.$$

If ν is absolutely continuous, $d\nu(x) = v(x)dx$ for some weight function v, then

$$[v]_{p,q,\alpha} := \sup_B \Big(v(B)\Big)^{1/q} |B|^{\alpha/n-1/p}.$$

We recall various criteria governing the trace inequality:

1. Maz'ya-Verbitsky, Adams and Hedberg (conditions in terms of Capacities) (involving the diagonal case $p = q$);
2. Maz'ya-Verbitsky (Pointwise conditions on v again in terms of I_α itself) (involving the diagonal case $p = q$);
3. E. Sawyer (Involving the operator (I_α itself, so-called Sawyer-type condition) (involving the diagonal case $p = q$).

Trace inequality is a special case of the two-weight inequality:

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q d\nu(x)\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx\right)^{1/p}. \tag{0.4}$$

Two-weight criteria for Riesz potentials:

1. E. Sawyer: (Involving the operator (I_α itself) ($p \leq q$);
2. M. Gabidzashvili and V. Kokilashvili (Transparent, integral-type conditions) ($p < q$)

   See e.g. the monograph [Kokilashvili-Krbec, 1993].

Trace Inequality

One of our aims: finding an appropriate Lorentz space $L^{p,s}$ such that the D. Adams-type condition for $p = q$:

$$\nu(B) \leq C|B|^{1-\alpha p/n} \tag{0.5}$$

on ν is both necessary and sufficient for the validity of the trace inequality

$$\|I_\alpha f\|_{L^p(\nu)} \leq C\|f\|_{L^{p,s}},$$

where I_α is the Riesz potential on $\mathbb{R}^n$.

Trace Inequality

To study this problem was motivated by the fact that the inequality fails for $p = s$, i.e. the inequality

$$\|I_\alpha f\|_{L^p(\nu)} \leq C\|f\|_{L^p},$$

under the condition (0.5).
As we have mentioned (see also the paper by)It is known that the latter condition is necessary and sufficient for the validity of

$$\|I_\alpha f\|_{L^q(\nu)} \leq C\|f\|_{L^p},$$

if and only if $p < q$ even when ν is absolutely continuous $d\nu(x) = v(x)dx$.

Trace Inequality

The trace inequality for Riesz potentials I_α implies the weighted estimate:

$$\|f\|_{L^q_v} \leq C\|\nabla f\|_{L^p}, \quad f \in C_0^\infty,\ d\nu(x) = v(x)dx, \tag{0.6}$$

which follows from the estimate

$$|f(x)| \leq CI_1(|\nabla f|)(x).$$

Here L^q_v is the weighted Lebesgue space defined with respect to the norm

$$\|f\|_{L^q_v} := \left(\int\limits_{\mathbb{R}^n} |f(x)|^q v(x)dx \right)^{1/q}.$$

In this direction it is known Fefferman–Phong inequality:

Theorem A.
Let $1 < p < \infty$ *and let* $0 < \alpha < n/p$. *Then the following inequality holds:*

$$\|I_\alpha f\|_{L^p_v} \le C\|f\|_{L^p}$$

for some $p < r$, *where*

$$[v]^*_{p,r,\alpha} := \sup_B |B|^{\alpha/n - 1/r} \left(\int_B v^{r/p}(x)dx \right)^{1/r} < \infty. \qquad (0.7)$$

Trace Inequality

It is easy to see that by Hölder's inequality we have that condition (0.7) is stronger than Adams type condition for $p = q$:

$$[v]_{p,\alpha} := [v]_{p,p,\alpha} := \sup_B \Big(v(B)\Big)^{1/p} |B|^{\alpha/n - 1/p}$$

in particular, $[v]_{p,\alpha} \leq [v]^*_{p,r,\alpha}$ for $r > p$.

Lorentz space

Let f be a measurable function on $\mathbb{R}^n$ and let $1 \le p < \infty$, $1 \le s \le \infty$. We say that f belongs to the Lorentz space $L^{p,s}$ if

$$\|f\|_{L^{p,s}} = \begin{cases} \left(s \int\limits_0^\infty \left(|\{x \in \mathbb{R}^n : |f(x)| > \tau\}| \right)^{s/p} \tau^{s-1} d\tau \right)^{1/s}, & \text{if } 1 \le s < \infty, \\ \sup\limits_{\tau > 0} \tau \Big(|\{x \in \mathbb{R}^n : |f(x)| > \tau\}| \Big)^{1/p}, & \text{if } s = \infty \end{cases}$$

is finite.

Lorentz space

If $p = s$, then $L^{p,s}$ coincides with the Lebesgue space L^p.
Denote by f^* a non-increasing rearrangement of f. Then by integration by parts it can be checked that (see also [Hunt]):

$$\|f\|_{L^{p,s}} = \begin{cases} \left(\frac{s}{p} \int\limits_0^\infty \left(t^{1/p} f^*(t) \right)^s \frac{dt}{t} \right)^{1/s}, & \text{if } 1 \leq s < \infty, \\ \sup\limits_{t>0} \{ t^{1/p} f^*(t) \}, & \text{if } s = \infty. \end{cases}$$

Lorentz space

Now we list some useful properties of Lorentz spaces (see, e.g., [Humt]):

1. $\|\chi_E\|_{L^{p,s}} = |E|^{1/p}$;
2. If $1 \le p < \infty$, $s_2 \le s_1$, then $L^{p,s_2} \hookrightarrow L^{p,s_1}$ with the embedding constant C_{p,s_1,s_2} depending only on p, s_1 and s_2;
3. positive constant $C_{p,s}$ such that

$$C_{p,s}^{-1}\|f\|_{L^{p,s}} \le \sup_{\|h\|_{L^{p',s'}}\le 1} \left| \int_X f(x)h(x)dx \right| \lesssim_{p,s} \|f\|_{L^{p,s}}$$

   for every $f \in L_w^{p,s}$, where $p' = p/(p-1)$, $s' = s/(s-1)$.
4. (Hölder's inequality) Let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. Then

$$\|f_1 f_2\|_{L^{p,s}} \leqslant C\|f_1\|_{L^{p_1,s_1}}\|f_2\|_{L^{p_2,s_2}}$$

   for all $f_1 \in L^{p_1,s_1}$ and $f_2 \in L^{p_2,s_2}$, where $C = C_{p,s,p_1,p_2,s_1,s_2}$.

Trace Inequality Characterization

Theorem

Let $1 < p < \infty$ and let $0 < \alpha < n/p$. Suppose that v is a non-negative locally integrable function on $\mathbb{R}^n$. Then the following statements are equivalent:

1. *there is a positive constant C such that for all $f \in L^{p,1}(\mathbb{R}^n)$,*

$$\|I_\alpha f\|_{L^p_v(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,1}(\mathbb{R}^n)} \tag{0.8}$$

2. *there is a positive constant c such that for all $f \in L^{p,1}(\mathbb{R}^n)$,*

$$\|M_\alpha f\|_{L^p_v(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,1}(\mathbb{R}^n)} \tag{0.9}$$

3. $$[v]_{p,\alpha} = \sup_B \big(v(B)\big)^{1/p} |B|^{\alpha/n - 1/p} < \infty.$$

Moreover, $[v]_{p,\alpha} \sim \|I_\alpha\| \sim \|M_\alpha\|$.

The case of an SHT

The Case of Spaces of Homogeneous Type

The case of an SHT

Let (X, d, μ) be a space of homogeneous type (SHT), i.e., X is an abstract set, and $d : X \times X \to \mathbb{R}_+$ is a quasi-metric satisfying the following conditions:

The case of an SHT

1. $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$ for all $x, y \in X$;
3. There is a constant $\kappa > 1$ such that $d(x,y) \leq \kappa\,(d(x,z) + d(z,y))$ for all $x, y, z \in X$,

and μ is a complete measure defined on a σ-algebra over X, such that the balls $B(x,r) := \{y \in X;\ d(x,y) < r\}$ are measurable with positive and finite μ-measure for all $x \in X$ and $r > 0$. Moreover, the doubling condition is satisfied:

$$\mu(B(x,2r)) \leq C_{DC}\,\mu(B(x,r)), \tag{0.10}$$

with a positive constant C_{DC} independent of x and r.

The case of an SHT

The best possible constant C_{DC} in (0.10) is called the *doubling constant*.

The case of an SHT

There exists many interesting examples of an SHT. We underline some of them:

1. Rectifiable regular curves (rectifiable curves satisfying G. David condition);
2. Homogeneous groups.

The case of an SHT

For a given SHT, (X, d, μ), and q satisfying $1 \leq q \leq \infty$, as usual, we denote by $L^q = L^q(X, \mu)$ the Lebesgue space equipped with the standard norm. Let $L^{p,s}(X, \mu)$ be the Lorentz space defined on (X, d, μ). If ν is another measure on X, then we denote the Lebesgue and Lorentz spaces defined with respect to ν by $L^q(X, \nu)$ and $L^{p,s}(X, \nu)$, respectively. If ν is absolutely continuous with respect to μ, i.e. $d\nu(x) = v(x)d\mu(x)$ for some μ-a.e. positive locally integrable function on X (i.e. v is a weight function on (X, d, μ)), then we use the symbols $L^q_v(X, \mu)$ and $L^{p,s}_v(X, \mu)$ for $L^q(X, \nu)$ and $L^{p,s}(X, \nu)$, respectively.

The case of an SHT

Define by $I_{\alpha,X} f$ the Riesz potential of a μ-measurable function f given by the formula:

$$I_{\alpha,X} f(x) = \int_X \mu(B_{xy})^{\alpha-1} f(y)\, d\mu(y), \quad x \in X,$$

where $0 < \alpha < 1$ and $B_{xy} := B(x, d(x, y))$.

The case of an SHT

Theorem

Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that (X, d, μ) is an SHT and ν is another measure on X. Then the inequality

$$\|I_{\alpha,X} f\|_{L^q(X,\nu)} \lesssim \|f\|_{L^p(X,\mu)}$$

holds if and only if

$$\sup_B \nu(B)^{1/q} \mu(B)^{\alpha - 1/p} < \infty,$$

where the supremum is taken over all balls B in X.

The case of an SHT

Theorem

Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Suppose that (X, d, μ) be an SHT. Assume that v is is a weight function on (X, d, μ). Then the following statements are equivalent:

(i) *there is a positive constant C such that for all $f \in L^{p,1}(X, \mu)$,*

$$\|I_{\alpha,X} f\|_{L^p_v(X,\mu)} \le C\|f\|_{L^{p,1}(X,\mu)}; \tag{0.11}$$

(ii)

$$[v]_{p,\alpha,X,\mu} := \sup_B \left(\int_B v(x)d\mu(x)\right)^{1/p} \mu(B)^{\alpha-1/p} < \infty.$$

Application for homogeneous groups

Let G be a homogeneous group, which is a nilpotent Lies group with homogeneous norm $r(\cdot)$, Haar measure dx and homogeneous dimention Q. Let

$$I_{\gamma,G}f(x) = \int_X r(xy^{-1})^{\gamma-Q} f(y)\, dy, \;\; x \in G, \;\; 0 < \gamma < Q,$$

be fractional integral operator on G.

Then the following statement holds:

Application for homogeneous groups

Theorem

Let $1 < p < \infty$ *and let* $0 < \gamma < Q/p$. *Then the following statements are equivalent:*

1. *there is a positive constant* C *such that for all* $f \in L^{p,1}(G)$,
$$\|I_{\gamma,G} f\|_{L^p_v(G)} \le C \|f\|_{L^{p,1}(G)}; \tag{0.12}$$
2. $$[v]_{p,\gamma,G} := \sup_B \left(\int_B v(x)dx \right)^{1/p} |B|^{\gamma/Q - 1/p} < \infty.$$

Moreover, $C \sim [v]_{p,\gamma,G}$., *where* C *is the the last inequality.*

This statement was proved in:

G. Imerlishvili and A. Meskhi, A note on the trace inequality for Riesz potentials. *Georgian Math. J.*, **28(5)** (2021), 739–745. doi: 10.1515/gmj-2020-207.

A. Meskhi, H. Rafeiro and S. Samko, Integral Operators in Non-Standard Function Spaces, A Sequel: Inequalities, Sharp Estimates, Bounded Variation, and Approximation, *Birkhäuser-Springer, To appear in 2026*.

It should be emphasized that this statement in Euclidean spaces was proved by another method by M. V. Korobkov and J. Kristensen..

Related References

M. A. Bhat and G. S. R. Kosuru, Wiener-Lebesgue Point Property for Sobolev Functions on Metric Spaces. *Mediterr. J. Math.* 21, 221 (2024).

F. de Almeida and L. S. M. Lima, Adams' trace principle in Morrey-Lorentz spaces on -Hausdorff dimensional surfaces. *Ann. Fenn. Math.,* 46(2):1161–1177, 2021.

L. Liu and J. Xiao. A trace law for the Hardy-Morrey-Sobolev space. J. Funct. Anal., 274(1), 2018.

V. Korobkov and J. Kristensen. The trace theorem, the Luzin N- and Morse-Sard properties for the sharp case of Sobolev-Lorentz mappings. *J. Geom. Anal.*, 28(3):2834–2856, 2018.

Part 2

Trace Inequality for one-sided fractional Integrals

Alexander Meskhi

One-sided Fractional Integrals. Diagonal case.

Now we present boundedness/compactness criteria for one-sided fractional integrals (Riemann-Liouville operator):

$$R_\alpha f(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$$

from $L^p(0,\infty)$ to $L^q_v(0,\infty)$ ($L^{q\infty}_v(0,\infty)$), where $0 < p,\ q < \infty$, $p > 1$ and $\alpha > \frac{1}{p}$.

Hardy Inequality

Theorem A. *Let* $1 \le p \le q < \infty$. *The inequality*

$$\left(\int_0^\infty \left|\int_0^x f(t)dt\right|^q v(x)dx\right)^{\frac{1}{q}} \le c\left(\int_0^\infty |f(x)|^p w(x)dx\right)^{\frac{1}{p}}, \qquad (2.1.1)$$

where the positive constant c *does not depend on* f, $f \in L^p_w(0,\infty)$, *holds if and only if*

$$A = \sup_{t>0}\left(\int_t^\infty v(x)dx\right)^{\frac{1}{q}}\left(\int_0^t w^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty \quad \left(p' = \frac{p}{p-1}\right).$$

Moreover, if c *is the best constant in* (2.1.1.), *then* $c \approx A$.

Compactness theorem for

Theorem C. *Let* (X,μ) *and* (Y,ν) *be* σ*–finite measure spaces and let* $1 < r, p < \infty$. *Then the condition*

$$M \equiv \| \| k(x,y) \|_{L_\nu^{r'}(Y)} \|_{L_\mu^p(X)} < \infty$$

implies the compactness of the operator

$$Kf(x) = \int_Y k(x,y) f(y) d\nu(y), \quad x \in X,$$

from $L_\nu^r(Y)$ *into* $L_\mu^p(X)$.

Trace Inequality for One-sided Fractional Integrals

Theorem 2.1.1. *Let $1 < p \leq q < \infty$, $\alpha > 1/p$. Then the following conditions are equivalent:*

(i) R_α is bounded from $L^p(0,\infty)$ into $L^q_v(0,\infty)$;

(ii) R_α is bounded from $L^p(0,\infty)$ into $L^{q\infty}_v(0,\infty)$;

(iii)

$$B \equiv \sup_{t>0} \Big(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \Big)^{1/q} t^{1/p'} < \infty;$$

(iv)

$$B_1 \equiv \sup_{k \in Z} \Big(\int_{2^k}^{2^{k+1}} v(x) x^{(\alpha - 1/p)q} dx \Big)^{1/q} < \infty.$$

Moreover, $\|R_\alpha\|_{L^p \to L^q_v} \approx \|R_\alpha\|_{L^p \to L^{q\infty}_v} \approx B \approx B_1$.

Proof. Denoting

$$I_{1\alpha}f(x) \equiv \int_0^{\frac{x}{2}} \frac{f(t)}{(x-t)^{1-\alpha}}dt$$

and

$$I_{2\alpha}f(x) \equiv \int_{\frac{x}{2}}^{x} \frac{f(t)}{(x-t)^{1-\alpha}}dt$$

for $f \in L^p(0,\infty)$ we write $R_\alpha f$ as

$$R_\alpha f(x) = I_{1\alpha}f(x) + I_{2\alpha}f(x).$$

for $f \in L^p(0,\infty)$ we write $R_\alpha f$ as

$$R_\alpha f(x) = I_{1\alpha} f(x) + I_{2\alpha} f(x).$$

We obtain

$$\|R_\alpha f\|^q_{L^q_v(0,\infty)} \le c_1 \int_0^\infty |I_{1\alpha} f(x)|^q v(x) dx + c_1 \int_0^\infty |I_{2\alpha} f(x)|^q v(x) dx = S_1 + S_2.$$

If $0 < t < \frac{x}{2}$, then $(x-t)^{\alpha-1} \le b x^{\alpha-1}$, where the positive constant b depends only on α. Consequently, using Theorem A with $w \equiv 1$, we have

$$S_1 \le c_2 \int_0^\infty \frac{v(x)}{x^{(1-\alpha)q}} \Big(\int_0^x |f(t)| dt \Big)^q dx \le c_3 B^q \|f\|^q_{L^p(0,\infty)}.$$

Next we shall estimate S_2. Using the Hölder inequality and the condition $1/p < \alpha$, we obtain

$$\|R_\alpha f\|^q_{L^q_v(0,\infty)} \le c_1 \int_0^\infty |I_{1\alpha} f(x)|^q v(x)dx + c_1 \int_0^\infty |I_{2\alpha} f(x)|^q v(x)dx = S_1 + S_2$$

If $0 < t < \frac{x}{2}$, then $(x-t)^{\alpha-1} \le bx^{\alpha-1}$, where the positive constant b depends only on α. Consequently, using Theorem A with $w \equiv 1$, we have

$$S_1 \le c_2 \int_0^\infty \frac{v(x)}{x^{(1-\alpha)q}} \left(\int_0^x |f(t)|dt \right)^q dx \le c_3 B^q \|f\|^q_{L^p(0,\infty)}.$$

Next we shall estimate S_2. Using the Hölder inequality and the condition $1/p < \alpha$, we obtain

$$S_2 = c_1 \int_0^\infty v(x) \Big| \int_{\frac{x}{2}}^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \Big|^q dx \le$$

$$\le c_1 \int_0^\infty v(x) \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt \Big)^{q/p} \Big(\int_{\frac{x}{2}}^x \frac{dt}{(x-t)^{(1-\alpha)p'}} \Big)^{q/p'} dx =$$

Boundedness criteria

$$= c_4 \sum_{k \in Z} \int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q+q/p'} \Big(\int_{\frac{x}{2}}^{x} |f(t)|^p dt \Big)^{q/p} dx \leq$$

$$\leq c_4 \sum_{k \in Z} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{q/p} \Big(\int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q+q/p'} dx \Big) \leq$$

$$\leq c_5 \sum_{k \in Z} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{q/p} \Big(\int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q} dx \Big) \cdot 2^{kq/p'} \leq$$

$$\leq c_5 B^q \sum_{k \in Z} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{q/p} \leq c_6 B^q \|f\|^q_{L^p(0,\infty)}.$$

which proves the sufficiency. Hence (iii) $\Rightarrow$ (i).

Boundedness criteria

Now we show that (ii) $\Rightarrow$ (iv). Let $f_k(x) = \chi_{(0,2^{k-1})}(x)$. Note that if $0 < y < 2^{k-1}$ and $x \in (2^k, 2^{k+1})$, then $(x-y)^{\alpha-1} \geq b_1 x^{\alpha-1}$, where the positive constant b_1 depends only on α. We have

$$\|R_\alpha f\|_{L_v^{q\infty}(0,\infty)} \geq c_7 \|\chi_{[2^k,2^{k+1})} x^{\alpha-1}\|_{L_v^{q\infty}(0,\infty)} 2^k \geq$$

$$\geq c_8 \|\chi_{[2^k,2^{k+1})}\|_{L_v^{q\infty}(0,\infty)} 2^{k\alpha} = c_8 \left(\int_{2^k}^{2^{k+1}} v(x)dx \right)^{1/q} 2^{k\alpha}.$$

On the other hand, $\|f_k\|_{L^p(0,\infty)} = c_9 2^{k/p}$ and by virtue of the boundedness of R_α from $L^p(0,\infty)$ into $L_v^{q\infty}(0,\infty)$ we find that $B_1 \leq \infty$.

Boundedness criteria

Next we prove that (iv) implies (iii). Let

$$B(t) \equiv \left(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{1/q} t^{1/p'}$$

and $t \in (0, \infty)$. Then $t \in (2^m, 2^{m+1}]$ for some $m \in Z$. We have

$$B(t)^q = \left(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right) t^{q/p'} \leq \left(\int_{2^m}^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right) 2^{\frac{(m+1)q}{p'}} =$$

$$= b_1 2^{mq/p'} \sum_{k=m}^\infty \left(\int_{2^k}^{2^{k+1}} \frac{v(x)}{x^{(1-\alpha)q}} dx \right) \leq$$

Boundedness criteria

$$b_1 2^{mq/p'} \sum_{k=m}^{\infty} 2^{-kq/p'} \int_{2^k}^{2^{k+1}} \frac{v(x)x^{q/p'}}{x^{(1-\alpha)q}} dx \leq$$

$$\leq b_2 B_1^q 2^{mq/p'} \sum_{k=m}^{\infty} 2^{-kq/p'} \leq b_3 B_1^q$$

and therefore $B \leq b_4 B_1 < \infty$.

As (i) implies (ii) we finally have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i). □

Compactness criteria

Theorem 2.1.5. *Let* $1 < p \le q < \infty$, $\alpha > \frac{1}{p}$. *Then the following conditions are equivalent:*

(i) R_α is compact from $L^p(0,\infty)$ into $L^q_v(0,\infty)$;

(ii) R_α is compact from $L^p(0,\infty)$ into $L^{q\infty}_v(0,\infty)$;

(iii)

$$B \equiv \sup_{t>0} \left(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{\frac{1}{q}} t^{\frac{1}{p'}} < \infty$$

and $\lim_{a\to 0} B^{(a)} = \lim_{b\to\infty} B^{(b)} = 0$, where

Compactness criteria

$$B^{(a)} \equiv \sup_{0<t<a} \left(\int_t^a \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{\frac{1}{q}} t^{\frac{1}{p'}};$$

$$B^{(b)} \equiv \sup_{t>b} \left(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{\frac{1}{q}} (t-b)^{\frac{1}{p'}};$$

(iv) $B < \infty$ and $\lim_{t\to 0} B(t) = \lim_{t\to\infty} B(t) = 0$, where

$$B(t) \equiv \left(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{\frac{1}{q}} t^{\frac{1}{p'}};$$

Compactness criteria

(v)

$$B_1 \equiv \sup_{k \in Z} \Big(\int_{2^k}^{2^{k+1}} v(x) x^{(\alpha - 1/p)q} dx \Big)^{\frac{1}{q}} < \infty$$

and $\lim_{k \to -\infty} B_1(k) = \lim_{k \to +\infty} B_1(k) = 0$, where

$$B_1(k) \equiv \Big(\int_{2^k}^{2^{k+1}} \frac{v(x)}{x^{(1/p - \alpha)q}} dx \Big)^{\frac{1}{q}}.$$

Compactness criteria

Proof. Let $0 < a < b < \infty$. We write $R_\alpha f$ as

$$R_\alpha f = \chi_{[0,a]} R_\alpha(f \cdot \chi_{(0,a)}) + \chi_{(a,b)} R_\alpha(f \cdot \chi_{(0,b)}) + \chi_{[b,\infty)} R_\alpha(f \cdot \chi_{(0,\frac{b}{2})}) +$$
$$+\chi_{[b,\infty)} R_\alpha(f \cdot \chi_{(\frac{b}{2},\infty)}) = P_{1\alpha} f + P_{2\alpha} f + P_{2\alpha} f + P_{4\alpha} f.$$

For $P_{2\alpha} f$ we have $P_{2\alpha} f(x) = \int_0^\infty k_1(x,y) f(y) dy$, with $k_1(x,y) = (x - y)^{\alpha-1} \chi_{(a,b)}(x)$ for $y < x$ and $k_1(x,y) = 0$ for $y \geq x$. Consequently

$$\int_0^\infty v(x) \Big(\int_0^\infty (k_1(x,y))^{p'} dy \Big)^{\frac{q}{p'}} dx \leq \Big(\int_a^b \frac{v(x)}{x^{(1-\alpha)q}} dx \Big) b^{\frac{q}{p'}} < \infty$$

and by Theorem C we conclude that $P_{2\alpha}$ is compact from $L^p(0,\infty)$ to $L^q_v(0,\infty)$.

Compactness criteria

In a similar manner we show that $P_{3\alpha}$ is compact, too.
Using Theorem 2.1.1 for the operators $P_{1\alpha}$ and $P_{4\alpha}$, we obtain

$$\|P_{1\alpha}\| \leq c_1 B^{(a)} \quad \text{and} \quad \|P_{4\alpha}\| \leq c_2 B^{(b/2)}.$$

Consequently

$$\|R_\alpha - P_{2\alpha} - P_{3\alpha}\| \leq \|P_{1\alpha}\| + \|P_{4\alpha}\| \leq c_1 B^{(a)} + c_2 B^{(b/2)} \to 0$$

as $a \to 0$ and $b \to \infty$.

Thus the operator R_α is compact, since it is a limit of compact operators. Hence (iii) $\Rightarrow$ (i).

Compactness criteria

Now we prove that (i) ⇒ (iv).

Note that the fact $B < \infty$ follows from Theorem 2.1.1. Thus we need to prove the remaining part. Let $f_t(x) = \chi_{(0,t)}(x)t^{-1/p}$. Then the sequence f_t is weakly convergent to 0. Indeed, assuming that $\varphi \in L^{p'}$, we obtain

$$\left| \int_0^\infty f_t(x)\varphi(x)dx \right| \leq \left(\int_0^t |\varphi(x)|^{p'} dx \right)^{\frac{1}{p'}} \to 0 \text{ as } t \to 0.$$

On the other hand, we have

$$\|R_\alpha f_t\|_{L^q_v(0,\infty)} \geq \left(\int_t^\infty v(x) \left(\int_0^t \frac{f_t(y)}{(x-y)^{1-\alpha}} dy \right)^q dx \right)^{\frac{1}{q}} \geq$$

$$\geq c_3 \left(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{\frac{1}{q}} t^{\frac{1}{p'}} = c_3 B(t).$$

$0 < \alpha < \frac{1}{p}$. Maz'ya-Verbitsky type conditions

Theorem 2.3.1. *Let* $1 < p < \infty$ *and let* $0 < \alpha < \frac{1}{p}$. *Then the inequality*

$$\int_0^\infty |R_\alpha f(x)|^p v(x)dx \le c_0 \int_0^\infty |f(x)|^p dx, \ \ f \in L^p(R_+), \tag{2.3.1}$$

holds if and only if $W_\alpha v \in L^{p'}_{loc}(R_+)$ *and*

$$W_\alpha[W_\alpha v]^{p'}(x) \le c W_\alpha v(x) \quad a.e. \tag{2.3.2}$$

Maz'ya-Verbitsky type conditions

First let us consider the integral equation

$$\varphi(x) = \int\limits_x^\infty \frac{\varphi^p(t)dt}{(t-x)^{1-\alpha}} + \int\limits_x^\infty \frac{v(t)}{(t-x)^{1-\alpha}}dt, \quad 0 < \alpha < 1, \qquad (1.14.1)$$

with given non–negative $v \in L_{loc}(R_+)$.

Recall that by R_α and W_α we denote the Riemann–Liouville and Weyl operators respectively, i.e.

$$R_\alpha f(x) = \int_0^x f(t)(x-t)^{\alpha-1}dt, \; x > 0, \; \alpha > 0,$$

$$W_\alpha f(x) = \int_x^\infty f(t)(t-x)^{\alpha-1}dt, \; x > 0, \; \alpha > 0.$$

Applications to Non-linear Integral Equations

Theorem 1.14.1. *Let* $1 < p < \infty$, $0 < \alpha < 1$, $p' = \frac{p}{p-1}$, *and* $A_p = (p'-1)(p')^{-p}$.

i) If $W_\alpha v \in L^p_{loc}(R_+)$ and the inequality

$$W_\alpha[W_\alpha v]^p(x) \le A_p W_\alpha v(x) \quad \text{a.e.} \tag{1.14.2}$$

holds, then (1.14.1) has a non-negative solution $\varphi \in L^p_{loc}(R_+)$. Moreover, $(W_\alpha v)(x) \le \varphi(x) \le p'(W_\alpha v)(x)$.

ii) If $0 < \alpha < \frac{1}{p'}$ and (1.14.1) has a non-negative solution in $L^p_{loc}(R_+)$, then $W_\alpha v \in L^p_{loc}(R_+)$ and

$$W_\alpha[(W_\alpha v)^p](x) \le c W_\alpha v(x) \quad \text{a.e.} \tag{1.14.3}$$

for some constant $c > 0$.

Applications to Non-linear Integral Equations

Proof. We shall use the following iteration procedure. Let $\varphi_0 = 0$, and let for $k = 0, 1, 2, \cdots$

$$\varphi_{k+1}(x) = W_\alpha(\varphi_k^p)(x) + W_\alpha v(x). \tag{1.14.4}$$

By induction it is easy to verify that

$$W_\alpha v(x) \leq \varphi_k(x) \leq \varphi_{k+1}(x), \quad k = 0, 1, 2, \cdots \tag{1.15.5}$$

From (1.14.4) we shall inductively derive an estimate of $\varphi_k(x)$.

Let

$$\varphi_k(x) \leq c_k W_\alpha v(x) \tag{1.14.6}$$

for some $k = 0, 1, \cdots$. It is obvious that $c_1 = 1$. Then (1.14.2), (1.14.3) and (1.14.5) yield

$$\varphi_{k+1}(x) \leq (A_p c_k^p + 1)(W_\alpha v)(x),$$

Applications to Non-linear Integral Equations

where A_p is the constant in (1.14.2). Thus $c_{k+1} = A_p c_k^p + 1$ for $k = 1, 2, \cdots$. Now by induction and the definition of A_p we deduce that the sequence $(c_k)_k$ is increasing. Indeed, it is obvious that $c_1 < c_2$. Let $c_k < c_{k+1}$. Then

$$c_{k+1} = A_p c_k^p + 1 < A_p c_{k+1} + 1 = c_{k+2}.$$

It is also clear that $(c_k)_k$ is bounded from the above by p' and consequently it converges. As the equation $z = A_p z^p + 1$ has only one solution, $x = p'$, it follows that $\lim_{k\to\infty} c_k = p'$. On the other hand, the sequence $(\varphi_k)_k$ is nondecreasing and by (1.14.6) we get

$$\varphi(x) = \lim_{k\to\infty} \varphi_k(x) \le p'(W_\alpha v)(x).$$

Applications to Non-linear Integral Equations

By our assumption $W_\alpha v \in L^p_{loc}(R_+)$ and from the preceding estimate we conclude that $\varphi \in L^p_{loc}(R_+)$. Moreover, $(W_\alpha v)(x) \leq \varphi(x) \leq p'(W_\alpha v)(x)$.

Now we prove the statement ii). Suppose (1.14.1) has a solution $\varphi \in L^p_{loc}(R^+)$. We have

$$W_\alpha(\varphi^p)(x) \leq \varphi(x) < \infty \quad \text{a.e.} \tag{1.14.7}$$

Hence $W_\alpha(\varphi^p) \in L^p_{loc}(R_+)$. Then from (1.14.7) we get

$$W_\alpha[W_\alpha(\varphi^p)(x)]^p(x) \leq W_\alpha(\varphi^p)(x) \quad \text{a.e.}$$

Applying Theorem 2.3.1, we deduce that

$$\|R_\alpha f\|_{L^{p'}_\rho} \leq \|f\|_{L^{p'}}, \tag{1.14.8}$$

Applications to Non-linear Integral Equations

where $\rho(x) = \varphi^p(x)$. But $(W_\alpha v)(x) \leq \varphi(x)$. Due to (1.14.7) we get

$$\|R_\alpha f\|_{L^{p'}_{\rho_1}} \leq c\|f\|_{L^{p'}} \tag{1.14.9}$$

with $\rho_1(x) = (W_\alpha v)^p(x)$. Using Lemma 2.3.2 we arrive at

$$\|R_\alpha f\|_{L^{p'}_v} \leq \|f\|_{L^{p'}}.$$

Applying Theorem 2.3.1 we come to condition (1.14.3). □

Two-weight criteria: off-diagonal case

For measurable $f : R \to R$ put

$$\mathcal{R}_\alpha f(x) = \int_{-\infty}^{x} (x-y)^{\alpha-1} f(y)\,dy,$$

$$\mathcal{W}_\alpha f(x) = \int_{x}^{+\infty} (y-x)^{\alpha-1} f(y)\,dy,$$

where $x \in R$ and $\alpha > 0$.

Two-weight criteria: off-diagonal case

Theorem 2.2.3. *Let* $1 < p < q < \infty$ *and* $0 < \alpha < 1$. *For the boundedness of* $\mathcal{R}_\alpha$ *from* $L^p_w(R)$ *into* $L^q_v(R)$ *it is necessary and sufficient that the conditions*

$$C_1 \equiv \sup_{x\in R, h>0} \Big(\int_{x-h}^{x+h} v(y)dy \Big)^{1/q} \Big(\int_{-\infty}^{x-h} \frac{w^{1-p'}(y)}{(x-y)^{(1-\alpha)p'}} dy \Big)^{1/p'} < \infty.$$

$$\sup_{a\in R, h>0} \Big(\int_{a-h}^{a+h} w^{1-p'}(y)dy \Big)^{1/p'} \Big(\int_{a+h}^{+\infty} \frac{v(y)}{(y-a)^{(1-\alpha)q}} dy \Big)^{1/q} < \infty.$$

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Alexander Meskhi
Lecture 2
Integral operators on non-homogeneous spaces

In the last two-three decades considerable attention of researchers was attracted to the study of problems of Harmonic Analysis such as the mapping properties of integral operators (Singular and fractional integral operators) defined on metric measure spaces with non-doubling measure (see the papers by [F. Nazarov, S. Treil, and A. Volberg], [X. Tolsa], [J. García– Cuerva and E. Gatto], [J. García– Cuerva and J. M. Martell], [V. Kokilashvili and A.M.], [T. Hytönen], etc.). Such spaces are called also non-homogeneous spaces.
The boundedness results for singular and fractional integral operators in Lebesgue spaces are mainly obtained under the "mild" growth condition on a measure.

Today we discuss similar problem for fractional integrals in Lorentz spaces, generally speaking, in multilinear setting and present recent results in this direction.

Fractional Integrals with measure. Boundedness Criteria

In this lecture we deliver boundedness and compactness triteria for fractiona integrals defined with respect to a measure in Lebesgue spaces.

Let (X, d, μ) be a nonhomogeneous space, i.e. a topological space endowed with a locally finite complete measure μ and quasi–metric $d : X \times X \to R^1$ satisfying only the conditions

Non-homogeneous spaces

(i) $d(x, x) = 0$ for all $x \in X$;

(ii) $d(x, y) > 0$ for all $x \neq y, x, y \in X$;

(iii) there exists a positive constant a_0 such that $d(x, y) \leq a_0 d(y, x)$ for every $x, y \in X$;

(iv) there exists a constant a_1 such that $d(x, y) \leq a_1(d(x, z) + d(z, y))$ for every $x,\ y,\ z \in X$;

(v) for every neighbourhood V of the point $x \in X$ there exists $r > 0$ such that the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in V;

(vi) the ball $B(x, r)$ is measurable for every $x \in X$ and for arbitrary $r > 0$;

Space of homogeneous type. Doubling Condition

We do not require the condition:

(vii) there exists a constant $b > 0$ such that $\mu B(x, 2r) \leq b\mu(B(x, r)) < \infty$ for every $x \in X$ and r, $0 < r < \infty$.

Space of homogeneous type. Doubling Condition

First we consider the integral operator

$$\mathcal{K}_0 f(x) = \int\limits_X (d(x,y))^{\gamma-1} f(y) d\mu, \quad 0 < \gamma < 1.$$

Theorem 6.1.1. *Let* $1 < p < q < \infty$ *and let* $0 < \gamma < 1$. *The operator* $\mathcal{K}_0$ *acts boundedly from* $L^p_\mu(X)$ *into* $L^q_\mu(X)$ *if only if there exists a constant* $c > 0$ *such that*

$$\mu B(x,r) \leq c r^\beta, \quad \beta = \frac{pq(1-\gamma)}{pq+p-q}, \tag{6.1.1}$$

for arbitrary balls $B(x,r)$.

To prove this theorem we need some results about the maximal function

$$\widetilde{M}f(x) = \sup_{r>0} \frac{1}{\mu B(x, N_0 r)} \int_{B(x,r)} |f(y)| d\mu,$$

where $N_0 = a_1(1 + 2a_0)$ and the constants a_0 and a_1 are from the definition of a quasi–metric.

Proposition 6.1.1. *$\widetilde{M}$ is bounded in $L^p_\mu(X)$, where $1 < p < \infty$.*

This can be proved by a well-known method using the following covering lemma.

Lemma A. *Suppose* E *is a bounded set (i.e.contained in a ball) in* X *such that for each* $x \in E$ *there is given a ball* $B(x, r(x))$. *Then there is a (finite or infinite) sequence of points* $x_j \in E$ *such that* $\{B(x_j, r(x_j))\}$ *is a disjoint family of balls and* $\{B(x_j, N_o r(x_j))\}$ *is a covering of* E.

For the proof Lemma A see [100], p. 15 (see also [56], p. 623).

Proof of Proposition 6.1.1. For $\lambda > 0$ we set

$$E_\lambda = \{x \in X : \widetilde{M} f(x) > \lambda\}.$$

Modified Maximal Operator

Let E be a bounded set. Then for arbitrary $x \in E_\lambda \cap E$ there exists a ball $B(x, r(x))$ such that

$$\frac{1}{\mu B(x, N_0 r(x))} \int_{B(x,r(x))} |f(y)| d\mu > \lambda.$$

By Lemma A, from the family $\{B(x, r(x))\}$ we can choose a finite or infinite sequence of balls such that

$$(E_\lambda \cap E) \subset \bigcup_{j=1}^{\infty} B(x_j, N_0 r(x_j)).$$

Space of homogeneous type. Doubling Condition

Further we obtain the estimates

$$\mu(E_\lambda \cap E) \le \sum_{j=1}^{\infty}(\mu B(x_j, N_0 r(x_j)) \le \lambda^{-1} \sum_{j=1}^{\infty} \int_{B(x_j, r(x_j))} |f(y)| d\mu \le$$

$$\le \lambda^{-1} \int_X |f(x)| d\mu.$$

Thus the weak type inequality

$$\mu\{x : \widetilde{M} f(x) > \lambda\} \le \lambda^{-1} \Big(\int_X |f(x)| d\mu \Big)$$

Space of homogeneous type. Doubling Condition

holds.

In addition, it is obvious that $\widetilde{M}$ has strong (∞, ∞) type. Finally using Marcinkiewicz's interpolation theorem we have the boundedness of $\widetilde{M}$ in $L^p_\mu(X)$ for $1 < p < \infty$.

Boundedness of Fractional Integral Operator

Proof of Theorem 6.1.1. *Necessity.* Let $\mathcal{K}_0$ be bounded from $L^p_\mu(X)$ to $L^q_\mu(X)$:

$$\left(\int_X |\mathcal{K}_0 f(x)|^q d\mu\right)^{1/q} \le c_3 \left(\int_X |f(x)|^p d\mu\right)^{1/p}.$$

In this inequality set $f = \chi_{B(a,r)}$, where $a \in X$ and $r > 0$. We have

$$\left(\int_{B(a,r)} \left(\int_{B(a,r)} (d(x,y))^{\gamma-1} d\mu\right)^q d\mu\right)^{1/q} \le c_3(\mu B(x,r))^{1/p}. \qquad (6.1.2)$$

When $x \in B(a,r)$ and $y \in B(a,r)$ it is obvious that $d(x,y) \le c_0 r$. Thus from (6.1.2) it follows that

$$r^{\gamma-1}\mu B(a,r)(\mu B(a,r))^{1/q} \le c_4(\mu B(a,r))^{1/p}.$$

When $x \in B(a,r)$ and $y \in B(a,r)$ it is obvious that $d(x,y) \leq c_0 r$. Thus from (6.1.2) it follows that

$$r^{\gamma-1}\mu B(a,r)(\mu B(a,r))^{1/q} \leq c_4(\mu B(a,r))^{1/p}.$$

From the last estimate we conclude that (6.1.1) holds.

Sufficiency. Let us introduce the notation

$$\Omega(x) \equiv \sup_{r>0} \frac{\mu B(x,r)}{r^{\beta}}.$$

For $x \in X$ and $r > 0$ represent $\mathcal{K}_0 f(x)$ by

$$\mathcal{K}_0 f(x) = \int\limits_{B(x,r)} d(x,y)^{\gamma-1} f(y) d\mu + \int\limits_{X \setminus B(x,r)} d(x,y)^{\gamma-1} f(y) d\mu \equiv$$

$$\equiv I_1(x) + I_2(x).$$

Boundedness of Fractional Integral Operator

Set $D_k = B(x, 2^{-k}r) \backslash B(x, 2^{-k-1}r)$. Then we have

$$|I_1(x)| \leq \sum_{k=0}^{\infty} \int_{D_k} (d(x,y))^{\gamma-1} |f(y)| d\mu \leq \sum_{k=0}^{\infty} 2^{-k\gamma} r^{\gamma} 2^k r^{-1} \times$$

$$\times \frac{\mu B(x, N_0 2^{-k} r)}{\mu B(x, N_0 2^{-k} r)} \int_{B(x, 2^{-k} r)} |f(y)| d\mu \leq$$

$$\leq c_5 r^{\gamma-1+\beta} \sum_{k=0}^{\infty} 2^{-k(\gamma-1+\beta)} \widetilde{M} f(x) \cdot \Omega(x) \leq c_6 r^{\gamma-1+\beta} \widetilde{M} f(x) \cdot \Omega(x).$$

Boundedness of Fractional Integral Operator

The last estimate holds because of the condition $\gamma - 1 + \beta > 0$. Therefore

$$|I_1(x)| \leq c_6 r^{\gamma-1+\beta} \Omega(x) \widetilde{M} f(x).$$

Now let $H_k = B(x, 2^{k+1}r) \backslash B(x, 2^k r)$. Hölder's inequality yields

$$|I_2(x)| \leq \|f\|_{L^p_\mu(X)} \left(\int_{X \backslash B(x,r)} (d(x,y))^{(\gamma-1)p'} d\mu \right)^{1/p'}.$$

Boundedness of Fractional Integral Operator

From the last inequality we have

$$|I_2(x)| \le \|f\|_{L^p_\mu(X)} \Big(\sum_{k=0}^{\infty} \int_{H_k} (d(x,y))^{(\gamma-1)p'} d\mu \Big)^{1/p'} \le$$

$$\le c_7 \|f\|_{L^p_\mu(X)} \Big(\sum_{k=0}^{\infty} 2^{k(\gamma-1)p'} r^{(\gamma-1)p'} \mu B(x, 2^{k+1}r) \Big)^{1/p'} \le$$

$$\le c_8 \|f\|_{L^p_\mu(X)} (\sum_{k=0}^{\infty} 2^{k(\gamma-1)p'} r^{(\gamma-1)p'} (2^{k+1}r)^{\beta})^{1/p'} \cdot (\Omega(x))^{1/p'}.$$

Hence

$$|I_2(x)| \leq c_8 \|f\|_{L^p_\mu(X)} r^{\gamma-1+\frac{\beta}{p'}} \Big(\sum_{k=0}^{\infty} 2^{k((\gamma-1)p'+\beta)} \Big)^{1/p'} (\Omega(x))^{1/p'} \leq$$
$$\leq c_9 \|f\|_{L^p_\mu(X)} r^{\gamma-1+\frac{\beta}{p'}} (\Omega(x))^{1/p'} . \tag{6.1.3}$$

The estimetes for I_1 and I_2 imply the following pointwise inequality:

$$|\mathcal{K}_0 f(x)| \leq c_{10} (r^{\gamma-1+\beta} \Omega(x) \widetilde{M} f(x) + $$
$$+ r^{\gamma-1+\beta/p'} (\Omega(x))^{1/p'} \|f\|_{L^p_\mu(X)}). \tag{6.1.4}$$

Boundedness of Fractional Integral Operator

Taking into account condition (6.1.1) and estimate (6.1.4), we deduce that

$$|\mathcal{K}_0 f(x)| \le c_{11}\big(r^{\gamma-1+\beta}\widetilde{M}f(x) + r^{\gamma-1+\beta/p'}\|f\|_{L^p_\mu(X)}\big)$$

for arbitrary $x \in X$ and $r > 0$.

In the last inequality we put

$$r = \|f\|^{\frac{p}{\beta}}\big(\widetilde{M}f(x)\big)^{-\frac{p}{\beta}}.$$

Thus we obtain the estimate

$$|\mathcal{K}_0 f(x)| \le c_{12}\|f\|^{\frac{(\gamma-1+\beta)p}{\beta}}_{L^p_\mu(X)}\big(\widetilde{M}f(x)\big)^{1-\frac{(\gamma-1+\beta)p}{\beta}}. \qquad (6.1.5)$$

From the definition of β we see that

$$q\left(1 - \frac{(\gamma - 1 + \beta)p}{\beta}\right) = p.$$

Using Proposition 6.1.1, from (6.1.5) we conclude that

$$\|\mathcal{K}_0 f\|_{L^q_\mu(X)} \le c_{13}\|f\|_{L^p_\mu(X)}.$$

The proof is complete.

Sobolev inequality

Corollary 6.1.1. *Let* $1 < p < \frac{1}{\gamma}$ *and* $\frac{1}{q} = \frac{1}{p} - \gamma$. *Then* $\mathcal{K}_0$ *acts boundedly from* $L^p_\mu(X)$ *into* $L^q_\mu(X)$ *if and only if*

$$\mu B(x,r) \leq c\, r.$$

This is a statement of Sobolev type for nonhomogeneous spaces.

Compactness

Compactness Criteria

In the theory of potential analysis and compact embeddings between function spaces, compact fractional integral operators play a significant role. We start our discussion about these operators with the classical compactness result, which states that if G is a bounded domain in $\mathbb{R}^n$, $p \leq q < \frac{pn}{n-\alpha p}$, then the Riesz potential operator

$$J_{\alpha,G} f(x) = \int_G \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad 0 < \alpha < n,$$

is compact from $L^p(G)$ to $L^q(G)$ (see, e.g., [12, p. 146]).

Compactness Criteria

Let (X, d, μ) be a quasi-metric measure space. Our aim is to provide a complete characterization of those measures μ for which the fractional integrals (or Riesz potentials) $T_{\gamma,\mu}$ associated with μ and a parameter $\gamma \in (0,1)$ given by the formula

$$T_{\gamma,\mu}f(x) := \int\limits_X \frac{f(y)}{d(x,y)^{1-\gamma}}\, d\mu(y), \quad f \in L^p(X,\mu), \quad x \in X$$

are compact operators from $L^p(X,\mu)$ into $L^q(X,\mu)$ for $1 < p < q < \infty$.

Compactness Criteria

As an application we obtain weighted variant of compactness criteria for Riesz potential operator $J_{\alpha,G}$, where G is a bounded domain of $\mathbb{R}^n$ and $0 < \alpha < n$, and for fractional integral operators defined on a rectifiable curve Γ in the complex plane $\mathbb{C}$:

$$K_{\alpha,\Gamma}\varphi(t) = \int\limits_{\Gamma} \frac{\varphi(\tau)}{|t-\tau|^{1-\alpha}}\, d\nu(\tau), \qquad \varphi \in L^p(\Gamma,\nu), \quad t \in \Gamma,$$

where $0 < \alpha < 1$ and ν is the arc-length measure on Γ.

Compactness Criteria

For such measure μ we define a family $\{\Omega_{\beta,\ell}\}_{\ell\in(0,d_X)}$ of positive and finite functions $\Omega_\ell\colon X \to (0,\infty)$ given by the formula:

$$\Omega_{\beta,\ell}(x) = \sup_{0<r\leq\ell} \frac{\mu(B(x,r))}{r^\beta}, \qquad x \in X,$$

which will play an important role in the proofs of the main results.

Compactness Criteria

THEOREM 3.1. *Let $0 < \gamma < 1 < p < q < \infty$ and let (X, d, μ) be a quasi-metric measure space with finite μ. Then, the integral operator $T_{\gamma,\mu}$ is compact from $L^p(X, \mu)$ to $L^q(X, \mu)$ if and only if μ is upper Ahlfors β-regular with β given by $(1-\gamma)/\beta = 1/p' + 1/q$, and*

$$\lim_{\ell \to 0} \sup_{x \in X} \Omega_{\beta,\ell}(x) = 0.$$

In the proof of the above Theorem we will use the following lemmas.

Compactness Criteria. Proof

LEMMA 3.1. *Let μ be a upper Ahlfors β-regular measure on a quasi-metric space (X, d, μ) with $\beta < (1-\gamma)r$ for some $\gamma \in (0,1)$ and $r > 0$. Then there is a positive constant $C = C(r,\gamma)$ such that, for all $\ell \in (0, d_X)$ and all $x \in X$,*

$$\int_{B(x,\ell)^c} d(x,y)^{(\gamma-1)r}\, d\mu(y) \le C\ell^{\beta+(\gamma-1)r}.$$

Compactness Criteria. Proof

P r o o f. For simplicity of notation, we put $\eta := (\gamma - 1)r$. Then, for any fixed $\ell \in (0, d_X)$ and $x \in X$, we have

$$I_\ell(x) := \int\limits_{B(x,\ell)^c} d(x,y)^{(\gamma-1)r}\, d\mu(y) = \int\limits_0^\infty \mu(E_{\ell,\lambda,x})\, d\lambda,$$

where for every $\lambda > 0$,

$$E_{\ell,\lambda,x} := \{y \in B(x,\ell)^c;\ d(x,y) < \lambda^{1/\eta}\}.$$

Compactness Criteria. Proof.

Observe that the set $E_{\ell,\lambda,x}$ is an empty set whenever $\lambda \geq \ell^\eta$. This implies that

$$I_\ell(x) = \int_0^{\ell^\eta} \mu(E_{\ell,\lambda,x})\, d\lambda.$$

Compactness Criteria. Proof

Clearly, $E_{\ell,\lambda,x} \subset B(x, \lambda^{1/\eta})$ and so by our hypothesis on μ there exists a constant $C_0 > 0$ such that

$$\mu(E_{\ell,\lambda,x}) \leq C_0 \lambda^{\beta/\eta}, \quad \lambda > 0.$$

Combining the above facts with $\beta/\eta > -1$ yields

Compactness Criteria. Proof.

$$I_\ell(x) \le C_0 \int\limits_0^{\ell^\eta} \lambda^{\beta/\eta}\, d\lambda = \frac{C_0}{\beta/\eta + 1} \ell^{\beta+\eta},$$

and this completes the proof. □

Compactness Criteria. Proof.

LEMMA 3.2. *Let μ be a upper Ahlfors β-regular measure on a quasi-metric space (X, d, μ) with $\beta < (1-\gamma)r$ for some $\gamma \in (0,1)$ and $r > 0$. Then there is a positive constant $C = C(r,\gamma)$ such that, for all ℓ and s satisfying the condition $0 < s < \ell < d_X$, and all $x \in X$,*

$$I_{\ell,s}(x) = \int\limits_{B(x,\ell)\setminus B(x,s)} d(x,y)^{(\gamma-1)r}\, d\mu(y) \leq C\,\Omega_{\beta,\ell}(x)\big(s^{(\gamma-1)r+\beta} + \ell^{(\gamma-1)r+\beta}\big).$$

Compactness Criteria. Proof.

P r o o f. Following the proof of Lemma 3.1 we let $\eta := (\gamma - 1)r$. Then, for any fixed $0 < s < \ell < d_X$ and $x \in X$, we have

$$I_{\ell,s}(x) = \int_0^{s^\eta} \mu\big((B(x,\ell) \setminus B(x,s)) \cap B(x,\lambda^{1/\eta})\big)\, d\lambda$$

$$= \int_{\ell^\eta}^{s^\eta} \mu\big((B(x,\ell) \setminus B(x,s)) \cap B(x,\lambda^{1/\eta})\big)\, d\lambda$$

Compactness Criteria. Proof.

$$+ \int_0^{\ell^\eta} \mu\big((B(x,\ell) \setminus B(x,s)) \cap B(x,\lambda^{1/\eta})\big)\, d\lambda$$

$$\leq \int_{\ell^\eta}^{s^\eta} \mu\big(B(x,\lambda^{1/\eta})\big)\, d\lambda + \int_0^{\ell^\eta} \mu\big(B(x,\ell)\big) d\lambda$$

$$=: f_{\ell,s}(x) + g_{\ell,s}(x).$$

Compactness Criteria. Proof.

Now we observe that $\lambda > \ell^\eta$ is equivalent to $\lambda^{1/\eta} \le \ell$ (by $\eta < 0$) and so

$$\mu\big(B(x, \lambda^{1/\eta})\big) \le \Omega_{\beta,\ell}(x)\lambda^{1/\eta}.$$

This estimate combined with $\beta/\eta > -1$ yields with $C = C(r, \gamma) > 0$,

$$f_{\ell,s}(x) \le C\,\Omega_{\beta,\ell}(x)s^{\beta+\eta}.$$

To finish it is enough to observe that

$$g_{\ell,s}(x) \le \ell^\eta \mu(B(x, \ell)) \le \ell^{\eta+\beta}\Omega_{\beta,\ell}(x).$$

□

Compactness Criteria. Proof.

Now we are ready to proof our main result.

Proof of Theorem 3.1. For simplicity of notation we put $T := T_{\gamma,\mu}$ and $L^r := L^r(X,\mu)$ for any $1 \leq r < \infty$. We have explained that the boundedness of $T_{\gamma,\mu}$ from L^p to L^q is equivalent to the Ahlfors β-regularity of μ. We prove sufficiency of the condition

$$\lim_{\ell \to 0} \sup_{x \in X} \Omega_{\beta,\ell}(x) = 0.$$

Compactness Criteria. Proof.

Fix $\ell \in (0, d_X)$. Following [6], for every $f \in L^p$, we decompose Tf as follows

$$Tf(x) = \int\limits_{B(x,\ell)} f(y)d(x,y)^{\gamma-1}d\mu(y) + \int\limits_{B(x,\ell)^c} f(y)d(x,y)^{\gamma-1}d\mu(y)$$

$$=: U_\ell f(x) + V_\ell f(x), \quad x \in X.$$

Observe that operator V_ℓ is an integral operator generated by kernel K_ℓ given by

$$K_\ell(x,y) := \chi_{B(x,\ell)^c}(y)d(x,y)^{\gamma-1}, \quad (x,y) \in X \times X.$$

Compactness Criteria. Proof.

We claim that $V_\ell \colon L^p \to L^q$ is compact for sufficiently small ℓ. To prove this, we show that $K_\ell \in L^{p'}[L^q]$ and then we apply Corollary 2.1. First we note that our hypothesis on μ implies that $\beta < (1-\gamma)p'$. Applying Lemma 3.1 with $r = p'$, we conclude that there exists a positive constant $C = C(p,q,\gamma)$ such that, for all $x \in X$ and $\ell \in (0, d_X)$,

$$\int_{B(x,\ell)^c} d(x,y)^{(\gamma-1)p'} \, d\mu(y) \leq C \ell^{(\gamma-1)p'+\beta}.$$

This proves that $K_\ell \in L^\infty[L^q] \subset L^{p'}[L^q]$ and so the claim is proved.

Compactness Criteria. Proof.

We will now estimate the norm of the operator U_ℓ. For any $0 \le f \in L^p$ with $\|f\|_{L^p} \le 1$ and all $x \in X$, we have

$$U_\ell f(x) = \sum_{k=0}^{\infty} \int_{B(x,2^{-k}\ell)\setminus B(x,2^{-(k+1)}\ell)} f(y) d(x,y)^{\gamma-1} d\mu(y)$$

$$\le C_\gamma \sum_{k=0}^{\infty} (2^{-k}\ell)^{\gamma-1} \frac{\mu(B(x,N_0 2^{-k}\ell))}{\mu(B(x,N_0 2^{-k}\ell))} \int_{B(x,2^{-k}\ell)} f(y) d\mu(y)$$

$$\le C_\gamma \ell^{\gamma-1+\beta} \widetilde{M}_\ell f(x) \Omega_{\beta,\ell}(x),$$

Compactness Criteria. Proof.

where $C_\gamma = 2^{1-\gamma}$ and

$$\widetilde{M}_\ell f(x) = \sup_{0<r<\ell} \frac{1}{\mu(B(x, N_0 r))} \int_{B(x,r)} |f| \, d\mu, \quad x \in X.$$

For a given $x \in X$ we put

$$r_x := \left(\widetilde{M}_\ell f(x)\right)^{-p/\beta} \Omega_{\beta,\ell}(x)^{-1/\beta}.$$

Compactness Criteria. Proof.

We consider two cases. Case (i): $\ell \le r_x$. Then we have that for some constant $C > 0$ which depends on p, q and γ,

$$U_\ell f(x) \le C\,\Omega_{\beta,\ell}(x)^{(1-\gamma)/\beta}\big(\widetilde{M}_\ell f(x)\big)^{1-\frac{p(\gamma-1+\beta)}{\beta}}$$
$$= C\,\Omega_{\beta,\ell}(x)^{(1-\gamma)/\beta}\big(\widetilde{M}_\ell f(x)\big)^{p/q}.$$

Case (ii): $\ell > r_x$. We represent $U_\ell f(x)$ as follows:

$$U_\ell f(x) = \int\limits_{B(x,r_x)} f(y)d(x,y)^{\gamma-1}\,d\mu(y) + \int\limits_{B(x,\ell)\setminus B(x,r_x)} f(y)d(x,y)^{\gamma-1}d\mu(y)$$
$$=: R_\ell f(x) + S_\ell f(x).$$

Compactness Criteria. Proof.

Further, repeating the arguments used above we deduce that there exists a constant $C = C_{p,q,\gamma} > 0$ such that

$$R_\ell f(x) \le C\,\Omega_{\beta,\ell}(x)^{(1-\gamma)/\beta}\big(\widetilde{M}_\ell f(x)\big)^{1-\frac{p(\gamma-1+\beta)}{\beta}}$$
$$= C\,\Omega_{\beta,\ell}(x)^{(1-\gamma)/\beta}\big(\widetilde{M}_\ell f(x)\big)^{p/q}.$$

Now we estimate $S_\ell f(x)$. By Lemma 3.2 with $s = r_x$, Hölder's inequality, the fact that $\beta+(\gamma-1)p' < 0$ and the condition $\|f\|_{L^p} \le 1$, for some $C > 0$, we have

Compactness Criteria. Proof.

$$
\begin{aligned}
S_\ell f(x) &\le \left(\int_{B(x,\ell)\setminus B(x,r_x)} d(x,y)^{(\gamma-1)p'} d\mu(y) \right)^{1/p'} \\
&\le C\Omega_{\beta,\ell}(x)^{1/p'} \left(r_x^{(\gamma-1)p'+\beta} + \ell^{(\gamma-1)p'+\beta} \right)^{1/p'} \\
&\le 2C\Omega_{\beta,\ell}(x)^{1/p'} r_x^{\gamma-1+\beta/p'} = 2C\Omega_{\beta,\ell}(x)^{(1-\gamma)/\beta} (\widetilde{M}_\ell f(x))^{p/q}.
\end{aligned}
$$

Hence,

$$
S_\ell f(x) \le 2C \big(\sup_{y\in X} \Omega_{\beta,\ell}(y) \big)^{(1-\gamma)/\beta} (\widetilde{M}_\ell f(x))^{p/q}.
$$

Compactness Criteria. Proof.

Consequently, there exists $C' > 0$ such that for all $x \in X$, we have

$$U_\ell f(x) \le C' \Big(\sup_{y \in X} \Omega_{\beta,\ell}(y) \Big)^{(1-\gamma)/\beta} \big(\widetilde{M}_\ell f(x) \big)^{p/q}.$$

Combining the above estimates, we conclude that there exists a constant $\widetilde{C} > 0$ such that

$$\|U_\ell\|_{L^p \to L^q} \le \widetilde{C} \Big(\sup_{x \in X} \Omega_{\beta,\ell}(x) \Big)^{(1-\gamma)/\beta}.$$

Thus we have proved that

$$\|T_{\gamma,\mu} - V_\ell\|_{L^p \to L^q} \le \|U_\ell\|_{L^p \to L^q} \le \widetilde{C} \Big(\sup_{x \in X} \Omega_{\beta,\ell}(x) \Big)^{(1-\gamma)/\beta}.$$

Compactness Criteria. Proof.

Therefore, our hypothesis yields that

$$\left\|T_{\gamma,\mu} - V_\ell\right\|_{L^p \to L^q} \to 0 \quad \text{as} \quad \ell \to 0.$$

Since $\{V_\ell\}_{\ell \in (0, d_X)}$ is a net of compact operators, $T_{\gamma,\mu}$ is necessarily compact. This completes the proof of sufficiency.

Now we prove necessity. Assume that $T_{\gamma,\mu} : L_p \to L_q$ is compact operator. We claim that $\lim_{\ell \to 0} \sup_{x \in X} \Omega_{\beta,\ell}(x) = 0$. Suppose the contrary. Then there exists a sequence (ℓ_n) with $\ell_n \to 0$ as $n \to \infty$ such that

$$\sup_{x \in X} \Omega_{\beta,\ell_n}(x) > 2\varepsilon, \quad n \in \mathbb{N}$$

Compactness Criteria. Proof.

for some $\varepsilon > 0$. This implies that there exist sequence (x_n) in X and real sequence (r_n) with $r_n \to 0$ as $n \to \infty$ (by $r_n \le \ell_n$) such that, for each $n \in \mathbb{N}$, we have

$$\frac{\mu(B(x_n, r_n))}{r_n^\beta} > \varepsilon. \qquad (*)$$

Consider a sequence (f_n) given for each $n \in \mathbb{N}$ by

$$f_n := \frac{\chi_{B(x_n, r_n)}}{\mu(B(x_n, r_n))^{1/p}}.$$

Compactness Criteria. Proof.

Clearly, $\|f_n\|_{L^p} = 1$ for each n. We claim that $f_n \to 0$ weakly in L^p as $n \to \infty$. In fact, for any functional x^* in $(L^p)^*$ there exists a function $g \in L^{p'}$ such that

$$x^*(f) = \int_X fg\, d\mu, \quad f \in L^p(X,\mu).$$

Combining Hölder's inequality with $\mu(B(x,r)) \leq Cr^\beta$ for all $x \in X$ and $r < d_X$ yields

$$|x^*(f_n)| \leq \|f_n\|_{L^p} \|\chi_{B(x_n,r_n)}\|_{L^{p'}} = \mu(B(x_n,r_n))^{1/p'} \leq Cr_n^{\beta/p'}.$$

Compactness Criteria. Proof.

Hence $f_n \to 0$ weakly in L^p as required. Since $T_{\gamma,\mu}$ is a compact operator, it follows that $T_{\gamma,\mu} f_n \to 0$ in L^q. To conclude observe that by $(*)$, we get that

$$\begin{aligned}
\|T_{\gamma,\mu} f_n\|_{L^q} &\geq \left(\int_{B(x_n,r_n)} \left(\int_{B(x_n,r_n)} \frac{f_n(y)}{d(x,y)^{1-\gamma}}\, d\mu(y) \right)^q d\mu(x) \right)^{1/q} \\
&\geq \frac{1}{(2\kappa r_n)^{1-\gamma}} \mu(B(x_n,r_n))^{1-\frac{1}{p}+\frac{1}{q}} \\
&= (2\kappa)^{\gamma-1} \left(\frac{\mu(B(x_n,r_n))}{r_n^\beta} \right)^{\frac{1-\gamma}{\beta}} \geq (2\kappa)^{\gamma-1} \varepsilon^{\frac{1-\gamma}{\beta}},
\end{aligned}$$

which is a contradiction. This completes the proof. □

Compactness Criteria. Proof.

To indicate some applications of Theorem 3.1, we need some further notation. Let G (resp., Γ) be a subset in $\mathbb{R}^n$ (resp., a rectifiable curve in the complex plain $\mathbb{C}$), then we let

$$G(x,r) := B(x,r) \cap G, \quad (\text{resp., } \Gamma(z,r) := D(z,r) \cap \Gamma),$$

where $B(x,r)$ (resp., $D(z,r)$) is a ball in $\mathbb{R}^n$ with radius $r > 0$ and center x (resp., disc in the complex plain $\mathbb{C}$ with radius r and center z).

Compactness Criteria. Proof.

If A is a measurable subset of a quasi-metric space (X, d, μ), then we consider A with the induced quasi-metric and with a measure μ restricted to the induced σ-algebra on A. In particular, balls in A have a form $A \cap B(x, r)$, where $B(x, r)$ is any ball in X. Consequently, we have that $G(x, r)$ and $\Gamma(z, r)$ are balls in $X = G$ and $X = \Gamma$ respectively.

We emphasize that basic mapping properties of fractional integral operators on Euclidean spaces defined with respect to the Lebesgue measure can be found, e.g., in the monograph [19].

Compactness Criteria. Proof.

THEOREM 3.2. *Let* $1 < p < q < \infty$ *and let* $n(1/p - 1/q) < \alpha < n$. *Suppose that* G *is a bounded domain in* $\mathbb{R}^n$ *and* v *is a weight function on* G. *Then the operator* $J_{\alpha,G}$ *is compact from* $L^p_{v^{1-p}}(G)$ *to* $L^q_v(G)$ *if and only if* μ *given by* $d\mu = v d\lambda_n$ *is upper Ahlfors* σ*-regular with* σ *defined by* $(n-\gamma)/\sigma = 1/p' + 1/q$, *and*

$$\lim_{\ell \to 0} \sup_{x \in G} \Omega_{\sigma,\ell}(x) = 0.$$

Compactness Criteria. Proof.

P r o o f. It is easy to see that the operator $J_{\alpha,G}$ is compact from $L^p_{v^{1-p}}(G)$ to $L^q_v(G)$ if and only if the operator

$$J_{\gamma,G,\mu}f(x) = \int_G \frac{f(y)}{d(x,y)^{1-\gamma}} d\mu(x)$$

is compact from $L^p(G,\mu)$ to $L^q(G,\mu)$ with $d\mu = v d\lambda_n$, $d(x,y) = |x-y|^n$ and $\gamma = \alpha/n$. By Theorem 3.1 we have that the latter compactness holds if and only if

Compactness Criteria. Proof.

$$\sup_{x\in G,\, r<d_G} \frac{\mu(\overline{G}(x,r))}{r^\eta} < \infty \quad \text{and} \quad \lim_{\ell\to 0} \sup_{x\in G} \overline{\Omega}_{\eta,\ell}(x) = 0,$$

where

$$\overline{\Omega}_{\eta,\ell}(x) := \sup_{r\leq \ell < d_G} \frac{\mu(\overline{G}(x,r))}{r^\eta}, \qquad x \in G$$

Compactness Criteria. Proof.

with

$$\frac{1-\gamma}{\eta} = \frac{1}{p'} + \frac{1}{q}, \quad \overline{G}(x,r) := \{y \in G;\ |x-y| < r^{1/n}\}.$$

To conclude it is enough to observe that for $R = r^n$ we have

$$\frac{\mu(\overline{G}(x,r))}{r^\eta} = \frac{\mu(G(x,R))}{R^{\frac{pq(n-\alpha)}{p+q-pq}}} = \frac{\mu(G(x,R))}{R^\sigma}.$$

□

Compactness Criteria. Proof.

THEOREM 3.3. *Let* $1 < p < q < \infty$ *and let* $1/p - 1/q < \alpha < 1$. *Suppose that* Γ *is a rectifiable curve on* $\mathbb{C}$ *and* v *is a weight function on* Γ. *Then the operator* $K_{\alpha,\Gamma}$ *is compact from* $L^p_{v^{1-p}}(\Gamma)$ *to* $L^q_v(\Gamma)$ *if and only if* μ *defined by* $d\mu = v d\nu$ *is upper Ahlfors* β*-regular with* β *given by* $(1-\alpha)/\beta = 1/p' + 1/q$, *and*

$$\lim_{\ell \to 0} \sup_{z \in \Gamma} \Omega_{\beta,\ell}(z) = 0.$$

Compactness Criteria. Proof.

We conclude with the following remark that our results could be applied to get sufficient conditions for compactness of positive operators from $L^p(X,\mu)$ into $L^q(X,\mu)$, which are dominated by discussed compact Riesz potential operators. For example, we shall associate to a given metric measure space (X,d,μ) a fractional kernel $K_\gamma \in L^0(\mu \times \mu)$ of type $\gamma \in (0,1)$ i.e., such that K_γ satisfies the following condition:

$$|K_\gamma(x,y)| \leq \frac{C}{d(x,y)^{1-\gamma}}$$

for all $x, y \in X$ with $x \neq y$ and some $C > 0$. The corresponding kernel K_γ generates a fractional integral operator given by

$$K_\gamma f(x) = \int_X k_\gamma(x,y) f(y)\, d\mu, \quad f \in L^p(\mu), \quad x \in X.$$

Compactness Criteria. Proof.

We conclude with the following remark that the sufficient condition for compactness of Riesz potential $T_{\gamma,\mu}$ from $L^p(X,\mu)$ into $L^q(X,\mu)$ for $1 < p < q < \infty$ assures the compactness of the fractional integral operator K_γ. To get this statement it is enough to apply a deep result due to Dodds and Fremlin [2] true even in setting of abstract Banach lattices, which states that if E and F are Banach lattices with E^* and F having order continuous norms and $S\colon E \to F$ is a positive operator dominated by a compact operator T (i.e., $0 \le S \le T$), then S is necessarily a compact operator.

Stein-Weiss Inequality on Non-homogeneous spaces

Let (X, ρ, μ) be a non-homogeneous space and let

$$I_\alpha f(x) = \int_X \frac{f(y)}{\rho(x,y)^{1-\alpha}}\, d\mu(y), \quad 0 < \alpha < 1.$$

Stein-Weiss Inequality on Non-homogeneous spaces

Theorem 3.1. *Let* $1 < p \le q < \infty$, $\frac{1}{p} - \frac{1}{q} \le \alpha < 1$, $\alpha \ne \frac{1}{p}$. *Suppose that* $\alpha p - 1 < \beta < p - 1$ *and* $\lambda = q\left(\frac{1}{p} + \frac{\beta}{p} - \alpha\right) - 1$. *Then the inequality*

$$\left(\int_X |I_\alpha f(x)|^q \rho(x_0, x)^\lambda \, d\mu(x)\right)^{\frac{1}{q}} \le c\left(\int_X |f(x)|^p \rho(x_0, x)^\beta \, d\mu(x)\right)^{\frac{1}{p}}, \qquad (1)$$

with the positive constant c *independent of* f *and* x_0, $x_0 \in X$, *holds if and only if*

$$B \equiv \sup_{a \in X,\, r > 0} \frac{\mu B(a, r)}{r} < \infty. \qquad (2)$$

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Trace Inequality for Fractional Integrals, and Related Topics

ALEXANDER MESKHI
Kutaisi International University
TSU A. Razmadze Mathematical Institute

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Fractional Integrals with measure

Introductions

In the last two-three decades considerable attention of researchers was attracted to the study of problems of Harmonic Analysis such as the mapping properties of integral operators (Singular and fractional integral operators) defined on metric measure spaces with non-doubling measure (see the papers by [F. Nazarov, S. Treil, and A. Volberg], [X. Tolsa], [J. García– Cuerva and E. Gatto], [J. García– Cuerva and J. M. Martell], [V. Kokilashvili and A.M.], [T. Hytönen], etc.). Such spaces are called also non-homogeneous spaces.
The boundedness results for singular and fractional integral operators in Lebesgue spaces are mainly obtained under the "mild" growth condition on a measure.

Today we discuss similar problem for fractional integrals in Lorentz spaces, generally speaking, in multilinear setting and present recent results in this direction.

Fractional Integrals with General Measure

In particular, we present a complete characterization of a measure μ guaranteeing the boundedness of the multilinear fractional integral operator $T^{(m)}_{\gamma,\mu}$ (defined with respect to a measure μ) from the product of Lorentz spaces $\prod_{k=1}^{m} L^{p_k}_{\mu}$ to the Lorentz space $L^{q}_{\mu}(X)$ are established. The results are new even for linear fractional integrals $T_{\gamma,\mu}$ (i.e., for $m = 1$). From the general results we have a criterion for the validity of Sobolev–type inequality in the multilinear setting.

Introduction. Fractional Integrals on Quasi-metric Measure Space

The potential operator defined on a quas-metric measure space (X, d, μ):

$$J_\gamma f(x) = \int\limits_X \frac{f(y)}{d(x,y)^{1-\gamma}} d\mu(y), \quad x \in X,$$

is a generalization of the Riesz potential

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad x \in \mathbb{R}^n.$$

Introduction. Bilinear Fractional Integrals

The bilinear version of I_α in $\mathbb{R}^n$ is the

$$B_\alpha(f,g)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}} dt, \quad 0 < \alpha < n.$$

The study of these fractional integrals was initiated by L. Grafakos (1992).

Introduction. Multilinear fractional Integrals

As a tool to understand B_α, the operators

$$\mathcal{I}_\alpha \overrightarrow{f}(x) = \int\limits_{\mathbb{R}^n} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}} d\overrightarrow{y}, \quad x \in \mathbb{R}^n,$$

written in the $m-$ linear form, where $0 < \alpha < nm$, $\overrightarrow{f} := (f_1, \cdots, f_n)$, $\overrightarrow{y} := (y_1, \cdots, y_n)$, were studied as well (see the paper by C. Kenig and E. Stein, 1999, Grafakos and Kalton, 2001).

Let (X, d, μ) be a quasi-metric measure space. The following operator is the generalization of I_α for a qiasi-metric measure space (X, d, μ):

$$T^{(m)}_{\gamma,\mu} \overrightarrow{f}(x) = \int\limits_{X^m} \frac{f_1(y_1) \cdots f_m(y_m) d\mu(\overrightarrow{y})}{\big(d(x, y_1) + \cdots + d(x, y_m)\big)^{m-\gamma}}, \quad 0 < \gamma < m, \quad x \in X.$$

Our aim is to characterize completely those measures measures ensuring the boundedness of $T^{(m)}_{\gamma,\mu}$ from $\prod_{j=1}^{m} L^{p_j}_{\mu}(X)$ to $L^{q}_{\mu}(X)$, where $\overrightarrow{f} := (f_1, \ldots, f_m)$, $d\mu(\overrightarrow{y}) := d\mu(y_1)\cdots d\mu(y_m)$, $L^{p_j}_{\mu}(X)$ and $L^{q}_{\mu}(X)$ are Lebesgue spaces defined on an (X, d, μ).

This result is new even for linear case $m = 1$. In particular, as a corollary, we have a complete characterization of a measure μ guaranteeing the boundedness of the fractional integral operator

$$T_{\gamma,\mu}g(x) = \int_X \frac{g(y)}{d(x,y)^{1-\gamma}} d\mu(y) \quad 0 < \gamma < 1, \ x \in X,$$

from $L^p_\mu(X)$ to $L^q_\mu(X)$ given by V. Kokilashvili and A. M, 2001.

We refer also to the papers by J. Garcia-Cuerva and J. M. Martell (2001), J. Garcia-Cuerva and A. E. Gatto (2004) for the Sobolev–type inequalities in the classical Lebesgue spaces for non-doubling measure.

Quasi-metric

Let (X, d, μ) be a topological space with a complete measure μ and a quasi-metric $d : X \times X \mapsto \mathbb{R}$ satisfying the conditions:
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) there is a positive constant κ such that for all $x, y, z \in X$,

$$d(x, y) \leq \kappa\big(d(x, z) + d(z, y)\big).$$

In the sequel we assume that all the balls $B(x, R)$ with center x and radius R are $\mu-$ measurable with finite measure, and that for every neighborhood V of $x \in X$, there exists $R > 0$ such that $B(x, R) \subset V$.
A measure μ is said to be Ahlfors upper β- regular if there is a positive constant b_0 such that

$$\mu(B(x, R)) \leq b_0 R^{\beta} \tag{0.1}$$

for all $x \in X$ and $R > 0$.

For a given quasi-metric measure space (X, d, μ) and q satisfying $1 < q < \infty$, as usual, we will denote by $L^q(\mu) = L^q(X, \mu)$ the Lebesgue space equipped with the standard norm. The weak space $L^{q,\infty}(\mu) := L^{q,\infty}(X, \mu)$ is to be the Banach space of all measurable functions f endowed with the quasi-norm

$$\|f\|_{L^{q,\infty}} = \sup_{\lambda>0} \lambda(\mu(\{x \in X;\ |f(x)| > \lambda\}))^{1/q}.$$

If $X = \mathbb{R}^n$ is equipped with the Lebseguae measure and $d\mu(x) = w(x)dx$, where w is a weight function, then we use the notation: $L^q(X, \mu) = L^q_w(\mathbb{R}^n)$ (resp $L^{q,\infty}(X, \mu) = L^{q,\infty}_w(\mathbb{R}^n)$).

he following statement was proved in [?] (see [?] for fractional integrals defined on Euclidean spaces).

Theorem

Let (X, d, μ) be a quasi-metric measure space and let $1 < r < q < \infty$, $0 < \gamma < 1$. Then $J_{\gamma,\mu}$ is bounded from $L^r(X, \mu)$ to $L^q(X, \mu)$ if and only if μ is upper Ahlfors β regular, where $\beta = \frac{(1-\gamma)rq}{rq+r-q}$.

From the previous statement it follows that (see [GHarcia-Cuerva nd Gatto, V. Kokilasjbili and A.M.]):

Corollary

If $1 < r < \infty$, $0 < \gamma < \frac{1}{r}$ and $q = \frac{r}{1-\gamma r}$, then $J_{\gamma,\mu}$ is bounded from $L^r(X, \mu)$ to $L^q(X, \mu)$ if and only if μ is upper Ahlfors 1-regular.

To prove Theorem 1, the authors use the boundedness of the modified Hardy-Littlewood maximal operator $\widetilde{\mathcal{M}}$ in $L^r(X, \mu)$ space with $1 < r < \infty$. In fact $\widetilde{\mathcal{M}}$ has the following property (see, e.g.,[Edmundas , Kokilashvili and Meshki, 2002], Ch. 6):

Proposition

The operator $\widetilde{\mathcal{M}}$ is weak $(1, 1)$ type and strong (r, r) type for $r > 1$.

To study mapping properties of fractional and singular integrals defined with respect to a non-doubling measure μ it is important to have the boundedness of modified Hardy-Littlewood maximal operator

$$\widetilde{\mathcal{M}}g(x) = \sup_{r>0} \frac{1}{\mu(B(x, N_0 r))} \int\limits_{B(x,r)} |g| \, d\mu,$$

where the constant N_0 depends only on a quasi-metric (see, e.g., [**?**, Chapters 6, 8] and references cited therein). To obtain the main results of this paper we use the boundedness of the following modified multi(sub)linear maximal operator:

$$\widetilde{M}\vec{f}(x) = \sup_{r>0} \prod_{j=1}^{m} \frac{1}{\mu(B(x, N_0 r))} \int_{B(x,r)} |f_j| \, d\mu, \tag{0.2}$$

with the same constant N_0 depending on a quasi-metric d.
This operator is a modification of the multi(sub)linear Hardy-Littlewood maximal operator

$$M\vec{f}(x) = \sup_{r>0} \prod_{j=1}^{m} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f_j| \, d\mu, \quad x \in X,$$

which turned out to be useful to control the multilinear Calderón-Zygmund operators and was introduced in [**?**] (see also [**?**]). A multi(sub)linear maximal operator M acts on the m- fold product of Lebesgue spaces and is smaller than the m-fold product of the Hardy–Littlewood maximal function. There it was used to obtain a precise control on multilinear singular integral operators of Calderón–Zygmund type and to build a theory of weights adapted to the multilinear setting.

Main Results

Theorem

Let (X, d, μ) be a quasi-metric measure space. Let $m \in \mathbb{N}$, $1 < p_j < \infty$ for each $j \in \{1, \ldots, m\}$. Suppose that $1 < p < q < \infty$ and that $0 < \gamma < 1$. Then the following statements about the operator $T_{\gamma,\mu}$ are equivalent:

(i) $T_{\gamma,\mu}$ *is bounded from* $L^{p_1}(X,\mu) \times \cdots \times L^{p_m}(X,\mu)$ *to* $L^q(X,\mu)$;

(ii) $T_{\gamma,\mu}$ *is bounded from* $L^{p_1}(X,\mu) \times \cdots \times L^{p_m}(X,\mu)$ *to* $L^{q,\infty}(X,\mu)$;

(iii) *The measure* μ *is upper Ahlfors* β*-regular with* $\beta = \frac{(m-\gamma)pq}{pqm+p-q}$.

Remark. Observe that conditions of Theorem 3 implies that $\beta > 0$.

Theorem 3 implies the following corollary.

Corollary

Let $m \in \mathbb{N}$, $1 < p_j < \infty$, $j \in \{1, \ldots, m\}$. *Suppose that* $0 < \gamma < \frac{1}{p}$ *and* $\frac{1}{p} - \frac{1}{q} = \gamma$. *Then the following statements are equivalent:*

(i) $T_{\gamma,\mu}$ *is bounded from* $L^{p_1}(X, \mu) \times \cdots \times L^{p_m}(X, \mu)$ *to* $L^q(X, \mu)$;

(ii) $T_{\gamma,\mu}$ *is bounded from* $L^{p_1}(X, \mu) \times \cdots \times L^{p_m}(X, \mu)$ *to* $L^{q,\infty}(X, \mu)$;

(iii) *The measure* μ *is upper Ahlfors* 1*-regular.*

Theorem 3 yields also the characterization of the following version of the weighted inequality for the multilinear potentials $I_\alpha \vec{f}$:

$$I_\alpha(\vec{f})(x) := \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}\, d\vec{y},$$

where $x \in \mathbb{R}^n$, $0 < \alpha < nm$, $d\vec{y} := dy_1 \cdots dy_m$.

Corollary

Let $m \in \mathbb{N}$, $1 < p_j < \infty$, $j \in \{1, \ldots, m\}$. Suppose that $1 < p < q < \infty$ and that $n\left[\frac{1}{p} - \frac{1}{q}\right] < \alpha < n$. Then the following statements are equivalent:

(i) *there is a positive constant C such that*

$$\|I_\alpha \vec{f}\|_{L^q_v(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{v^{1-p_j}}(\mathbb{R}^n)};$$

(ii) *there is a positive constant c such that*

$$\|I_\alpha \vec{f}\|_{L^q_v(\mathbb{R}^n)} \le c \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\infty}_{v^{1-p_j}}(\mathbb{R}^n)};$$

(iii) *there exists a positive constant C such that for all $a \in \mathbb{R}^n$ and $r > 0$,*

$$\int_{B(a,r)} v(y)\, dy \le C r^{\frac{(mn-\alpha)pq}{pqm+p-q}}.$$

Linear case

Theorem

Let (X, d, μ) be a quasi-metric measure space, $0 < \gamma < 1$. Let $1 < r < \frac{1}{\gamma}$, and $\frac{1}{p} = \frac{1}{r} - \gamma$. Let s and q be such that $\frac{r}{p} = \frac{s}{q}$. Then inequality

$$\|T_{\gamma,\mu}g\|_{L^{p,q}_{\mu}(X)} \leq C\|g\|_{L^{r,s}_{\mu}(X)},$$

with the positive constant C independent of g holds if and only if

$$\mu B(x, R) \leq b_0 R.$$

Linear case

Taking a Borel measure ν on $\mathbb{R}^n$ and fractional integral

$$I_{\alpha,\nu} f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, d\nu(y), \quad x \in \mathbb{R}^n,\ 0 < \alpha < n,$$

as a corollary we have the following characterization of the Sobolev inequality in Lorents spaces defined on $\mathbb{R}^n$ with respect to ν:

Theorem

Let $0 < \alpha < n$ and let $1 < r < \frac{n}{\alpha}$. We set

$$p = \frac{nr}{n - \alpha r}.$$

Let s and q be such that $\frac{r}{p} = \frac{s}{q}$. Then inequality

$$\|I_{\alpha,\nu} g\|_{L^{p,q}_{\nu}(\mathbb{R}^n)} \le C \|g\|_{L^{r,s}_{\nu}(\mathbb{R}^n)},$$

with the positive constant C independent of g holds if and only if

Proof of the main statement

We prove the theorem in the case $\mu(X) = \infty$. The proof for $\mu(X) < \infty$ is similar. Taking the test functions $\vec{f}_{a,r} = (f_1^{(a,r)}, \dots, f_m^{(a,r)})$, where $f_j^{(a,r)} = \chi_{B(a,r)}$ for each $j \in \{1, \dots, m\}$, we have that

$$\prod_{j=1}^{m} \|f_j\|_{L^{p_j}(X,\mu)} = \big(\mu(B(a,r))\big)^{1/p}.$$

On the other hand, since $T_{\gamma,\mu}\vec{f}(x) \geq C_{\gamma,\gamma} r^{\gamma-m}\big(\mu(B(a,r))\big)^m$ for $x \in B(a,r)$, by using the boundedness of $T_{\gamma,\mu}$ from $\prod_{j=1}^m L^{p_j}(X,\mu)$ to $L^{q,\infty}(X,\mu)$, we have that (ii) $\Rightarrow$ (iii). Since (i) $\Rightarrow$ (ii) it remains to show that (iii) $\Rightarrow$ (i). Without loss of generality we assume that $0 \leq f_j \in L^{p_j}(X,\mu)$ for each $1 \leq j \leq m$. We introduce the notation:

$$k(x, y_1, \dots, y_n) := d(x, y_1) + \dots + d(x, y_m);$$

$$\Omega(x) := \sup_{r>0} \frac{\mu(B(x,r))}{r^\beta}. \tag{0.3}$$

Proof

We have

$$T_{\gamma,\mu}\vec{f}(x) = C_{m,\gamma} \int\limits_{X^m} f_1(y_1)\cdots f_m(y_m) \int\limits_{k(x,y_1,\ldots,y_m)}^{\infty} t^{\gamma-m-1}\,dt\,d\mu(\vec{y})$$

$$= C_{m,\gamma} \int\limits_0^{\infty} t^{\gamma-m-1} \int\limits_{\{\vec{y}:k(x,y_1,\ldots,y_m)<t\}} f_1(y_1)\cdots f_m(y_m)\,d\mu(\vec{y})\,dt$$

$$\leq C_{m,\gamma} \int\limits_0^{\infty} t^{\gamma-m-1} \prod_{j=1}^{m} \int\limits_{B(x,t)} f_j(y_j)d\mu(y_j)\,dt$$

$$= C_{m,\gamma} \int\limits_0^{r} t^{\gamma-m-1} \prod_{j=1}^{m} \int\limits_{B(x,t)} f_j(y_j)d\mu(y_j)\,dt$$

$$+ C_{m,\gamma} \int^{\infty} t^{\gamma-m-1} \prod^{m} \int f_j(y_j)d\mu(y_j)\,dt$$

Proof

Now we estimate $I_1(x,r)$ and $I_2(x,r)$ separately. For that we use the trick similar to that of L. Hedberg [**?**]. We have

$$I_1(x,r) \le C_{m,\gamma} \widetilde{M}\ \vec{f}(x) \int\limits_0^r t^{\gamma-m-1} \Big(\mu(B(x,N_0 t)\Big)^m dt$$

$$\le C_{m,\gamma} \widetilde{M}\ \vec{f}(x) \Big(\Omega(x)\Big)^m \int\limits_0^r t^{\gamma-m-1+m\beta}\, dt,$$

where $\widetilde{M}$ is the modified multi(sub)linear Hardy-Littlewood maximal operator defined by formula (0.2).
Here we used the estimate

$$\mu(B(x,t) \le \Omega(x) t^{\beta},$$

where $\Omega(x)$ is defined above

Proof

Observe now that $\gamma - m - 1 + m\beta > -1$ because $p < q$. Consequently, taking into account the obtained estimate above, we conclude that

$$I_1(x,r) \le C_{m,\gamma} \widetilde{M}\, \vec{f}(x) \Big(\Omega(x)\Big)^m r^{\gamma - m + m\beta}. \tag{0.4}$$

Further, by applying Hölder's inequality we have that

$$I_2(r,x) \le C_{m,\gamma} \int_r^\infty t^{\gamma-m-1} \prod_{j=1}^m \Big(\mu(B(x,t))\Big)^{1/p_j'} \Bigg(\int_{B(x,r)} f_j^{p_j}(y_j) d\mu(y_j) \Bigg)^{1/p_j} dt$$

$$\le C_{m,\gamma} \Bigg(\int_r^\infty t^{\gamma-m-1} \Big(\mu(B(x,t))\Big)^{m-1/p} dt \Bigg) \prod_{j=1}^m \|f_j\|_{L^{p_j}(X,\mu)}$$

$$\le C_{m,\gamma} \Big(\Omega(x)\Big)^{m-1/p} \Bigg(\int_r^\infty t^{\gamma-m-1+\beta(m-1/p)} dt \Bigg) \prod_{k=1}^m \|f_j\|_{L^{p_j}(X,\mu)}$$

Proof

In the last equality we used the fact that $\gamma - m + \beta(m - 1/p) < 0$. This fact holds because $p > 1$. Summarizing these estimates we find that

$$T_{\gamma,\mu}\vec{f}(x) \le C_{\gamma,\mu}\Big(\widetilde{M}\ \vec{f}(x)\big(\Omega(x)\big)^{m} r^{\gamma-m+m\beta} + \big(\Omega(\vec{x})\big)^{m-1/p} r^{\gamma-m+m\beta-\beta/p}\prod_{j=1}^{m}\|f \tag{0.5}$$

Proof

Taking

$$r := \left(\widetilde{M}\ \vec{f}(x)\right)^{-p/\beta}\left(\prod_{j=1}^{m}\|f_k\|_{L^{p_j}(X,\mu)}\right)^{p/\beta}$$

in the above inequality and using the condition $\sup_{x\in\Omega}\Omega(x) < \infty$, we find that

$$T_{\gamma,\mu}\vec{f}(x) \le C_{\gamma,\mu}\left(\widetilde{M}\ \vec{f}(x)\right)^{1-\frac{p}{\beta}(\gamma-m+\beta m)}\left(\prod_{j=1}^{m}\|f\|_{L^{p_j}(X,\mu)}\right)^{\frac{p}{\beta}(\gamma-m+\beta m)}. \tag{0.6}$$

Proof

Combining Proposition regarding the boundedness of $\widetilde{M}$ with $q\left[1-\frac{p}{\beta}(\gamma-m+\beta m)\right]$ yields

$$
\begin{aligned}
\|T_{\gamma,\mu}\vec{f}\|_{L^q(X,\mu)} &\le C_{\gamma,\mu}\|\widetilde{M}(\vec{f})^{1-\frac{p}{\beta}(\gamma-m+\beta m)}\|_{L^q(X,\mu)}\left(\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(X,\mu)}\right)^{\frac{p}{\beta}(\gamma-m+\beta} \\
&= C_{\gamma,\mu}\|\widetilde{M}\vec{f}\|^{p/q}_{L^p(X,\mu)}\left(\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(X,\mu)}\right)^{\frac{p}{\beta}(\gamma-m+\beta m)} \\
&\le C_{\gamma,\mu}\prod_{j=1}^{m}\|f_j\|^{p/q}_{L^{p_j}(X,\mu)}\prod_{j=1}^{m}\|f_j\|^{\frac{p}{\beta}(\gamma-m+\beta m)}_{L^{p_j}(X,\mu)} = C_{\gamma,\mu}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(X,\mu}
\end{aligned}
$$

□

Now we observe that Corollary 5 follows from Theorem 3 by taking $X=\mathbb{R}^n$, $d\mu(x)=v(x)dx$, $d(x,z)=|x-z|^n$ there. The condition $n\left[\frac{1}{p}-\frac{1}{q}\right]<\alpha<n$ guarantees that $0<\beta<n$.

October 25, 2025

Olsen's inequality. Sharp Estimates
Lecture 3

Olsen Inequality

In the literature Classical Olsen's inequality is called the following bilinear inequality:

$$\|g(I_\alpha f)\|_X \le C\|g\|_E\|f\|_F, \tag{0.1}$$

where X, E and F are Morrey spaces (generally speaking, different) and I_α is the Riesz potential (fractional integral) operator:

$$I_\alpha(f)(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad 0 < \alpha < n, \quad x \in \mathbb{R}^n.$$

Introduction

Inequalities of type (0.1) play an important role in the study of perturbed Schrödinger equation (see Olsen, 1995). We refer to the papers by Y. Sawano, S. Sugano, and H. Tanaka for subsequent improvements of Olsen's original inequality and applications.

Introduction

Our aim is:

1) to establish (0.1) in the sharp form, i.e. the Morrey norm $\left\|g\right\|_E$ in (0.1) be optimal (can not be replaces by the smaller norm) (it is connected with the trace problem);

2) to derive such an optimal inequality (0.1) for multilinear fractional integrals.

Perturbed Schrödinger equation

This inequality comes from the study of solvability of the estimation of the solution of Perturbed Schrödinger equation (see Olsen 1995).

Perturbed Schrödinger equation

This inequality comes form the estimation of the solution of

$$\Delta u(x) + V(x)u(x) = 0, \quad u(x)\Big|_{\partial\Omega} = 1, \tag{0.2}$$

where $\Omega = \{x : x \in \mathbb{R}^3, |x| < R\}$.
By substituting $v(x) = u(x) - 1$ in (0.2), we have

$$\Delta v(x) + v(x)V(x) = -V(x), \quad v(x)\Big|_{\partial\Omega} = 0. \tag{0.3}$$

Perturbed Schrödinger equation

Applying the inverse Dirichlet Laplacian to both sides of (0.3) we get

$$(\Delta + V)v = -V \Rightarrow (1 + \Delta^{-1})v = -\Delta^{-1}V \Rightarrow v =$$

$$= (1 - (-\Delta^{-1}v))^{-1}(-\Delta^{-1}V)$$

$$\Rightarrow v = \sum_{n=1}^{\infty} \Big(-\Delta^{-1}V\Big)^{n}.$$

Taking again $u(x) = v(x) + 1$ we find that

$$u = 1 + \sum_{n=1}^{\infty} \Big(-\Delta^{-1}V\Big)^{n}.$$

Let $G(x, y)$ be Green's function for Dirichlet Laplacian $-\Delta$. Then

$$u = 1 + \int_\Omega G(x, x_1)V(x_1)dx_1 + \int_\Omega G(x, x_1)V(x_1) \int_\Omega G(x_1, x_2)V(x_2)dx_2dx_1 + \cdots$$

Perturbed Schrödinger equation

Since

$$|G(x,y)| \leq \frac{1}{4\pi}\frac{1}{|x-y|},$$

we have

$$|u(x)| \leq 1 + C\int\limits_{\Omega}\frac{V(x_1)}{|x-x_1|}dx_1 +$$

$$C^2\int\limits_{\Omega}\frac{V(x_1)}{|x-x_1|}\int\limits_{\Omega}\frac{V(x_2)}{|x_1-x_2|}dx_2\,dx_1 + \cdots .$$

Perturbed Schrödinger equation

Morrey space M_t^p, $1 \le p \le t < \infty$ is defined with respect to the norm:

$$\|f\|_{M_q^p} = \sup_Q |Q|^{\frac{1}{t}-\frac{1}{p}} \left(\int_Q |f(x)|^p dx \right)^{1/p}.$$

If $p = t$, then we have the classical Lebesgue space L^p.

Olsen studied the problem when V is in the Morrey space $M_{3/2}^p$ which turned out to be right space by the following reason:
Call the scaled function $u_R(x) = u(Rx)$. Then it satisfies the equation

$$\Delta u_R(x) + R^2 V(Rx) u_R(x) = 0.$$

for $|x| \le 1$ with the boundary condition $u_R(x) = 1$ for $|x| = 1$. Let $V_R(x) = R^2 V(Rx)$. A reasonable requirement for a norm on the potential V is for the norm to be invariant under the transformation $V \mapsto V_R$. The spaces $M_{3/2}^p$ has this invariance property.

Perturbed Schrödinger equation

Olsen proved that if $V \in M^p_{3/2}$, u is the solution of

$$\Delta u(x) + V(x)u(x) = 0, \quad u(x)\Big|_{\partial\Omega} = 1,$$

$v(x) = u(x) - 1$, $1 < p < 3/2$, $1 < s \le t < \infty$, $s < 2pt/3$, then

$$\|v\|_{M^s_t} \le C_1 \|V\|_{M^p_{3/2}} |\Omega|^{1/t}$$

and

$$\left(\frac{1}{|\Omega|}\int_\Omega |v|^s\right)^{1/s} \le C_2 \|V\|_{M^p_{3/2}},$$

if $\|V\|_{M^p_{3/2}} < \varepsilon$ with sufficiently small ε.

Perturbed Schrödinger equation

In the proof of the theorem it is important to use the sharp estimate for the mapping

$$V(\cdot) \mapsto V(\cdot) \int\limits_{\Omega} \frac{V(y)}{|\cdot - y|} dy$$

in Morrey spaces.

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad x \in \mathbb{R}^n.$$

The bilinear version of I_α in $\mathbb{R}^n$ is the

$$B_\alpha(f,g)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}} dt, \quad 0 < \alpha < n.$$

The study of these fractional integrals was initiated by L. Grafakos (1992).

As a tool to understand B_α, the operators

$$\mathcal{I}_\alpha \overrightarrow{f}(x) = \int\limits_{\mathbb{R}^n} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}} d\overrightarrow{y}, \quad x \in \mathbb{R}^n,$$

written in the $m-$ linear form, where $0<\alpha<nm$, $\overrightarrow{f} := (f_1, \cdots, f_n)$, $\overrightarrow{y} := (y_1, \cdots, y_n)$, were studied as well (see the paper by C. Kenig and E. Stein, 1999).

Abstract

We established a sharp Olsen type inequality

$$\left\| g\,\mathcal{I}_\alpha(f_1,\dots,f_m) \right\|_{L_r^q} \le C \left\| g \right\|_{L_\ell^q} \prod_{j=1}^{m} \left\| f_j \right\|_{L_{s_j}^{p_j}}$$

for multilinear fractional integrals, where L_r^q, L_ℓ^q, $L_{s_j}^{p_j}$, $j = 1, \dots, m$, are Morrey space with indices satisfying certain homogeneity conditions. This inequality is sharp because it gives necessary and sufficient condition on weights function V for which the inequality

$$\left\| \mathcal{I}_\alpha(f_1,\dots,f_m) \right\|_{L_r^q(V)} \le C \prod_{j=1}^{m} \left\| f_j \right\|_{L_{s_j}^{p_j}}$$

holds.

Abstract

We also derive a characterization of the trace inequality

$$\left\| B_\alpha(f_1, f_2) \right\|_{L^q_r(d\mu)} \leq C \prod_{j=1}^{2} \left\| f_j \right\|_{L^{p_j}_{s_j}(\mathbb{R}^n)},$$

in terms of a Borel measure μ, where B_α is the bilinear fractional integral operator given by the formula
$B_\alpha(f_1, f_2)(x) = \int\limits_{\mathbb{R}^n} \frac{f_1(x+t) f_2(x-t)}{|t|^{n-\alpha}} dt, \quad 0 < \alpha < n,$
Some of our results are new even in the linear case, i.e. when $m = 1$.

Weight results for $\mathcal{I}_\alpha$ in Lebesgue spaces

For the multilinear fractional operator $\mathcal{I}_\alpha$ and Moen (2009) obtained one-weight criteria, as well as “power bump” conditions for the two–weight inequalities. Various type of one and two–weight multilinear problems for these operators in Lebesgue spaces were studied by many authors (see e.g. the papers by Moen, Pradolini, Chen and Xue, Sawano, Nakai, Mastylo, Kokilashvili and A.M. , Shi and Tao etc)

Morrey Spaces

Let $1 \leq q \leq r < \infty$ and let $d\mu$ be a Borel measure on $\mathbb{R}^n$. We denote by $L^q_r(d\mu)$ the Morrey space of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^q_r(d\mu)} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^{\frac{1}{q}-\frac{1}{r}}} \left(\int_Q |f(x)|^q d\mu(x) \right)^{1/q} < \infty. \qquad (0.4)$$

If V is a locally integrable a.e. positive function on $\mathbb{R}^n$, i.e. a weight on $\mathbb{R}^n$, then we denote $L^q_r(d\mu)$ by $L^q_r(V)$.

Fractional Integrals in Morrey Spaces: Known Results

Proposition A. (Spanne, unpublished) *Let* $0 < \alpha < n$, $1 < p_0 \le s_0 < \infty$, $1 < q_0 \le r_0 < \infty$. *Suppose that* $\frac{1}{s_0} - \frac{1}{r_0} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$. *Then* I_α *is bounded from* $L^{p_0}_{s_0}$ *to* $L^{q_0}_{r_0}$.

Proposition B. (Adams, 1975) *Let* $0 < \alpha < n$, $1 < p_0 \le s_0 < \infty$, $1 < q_0 \le r_0 < \infty$. *Suppose that* $\frac{1}{r_0} = \frac{1}{s_0} - \frac{\alpha}{n}$, $\frac{q_0}{r_0} = \frac{p_0}{s_0}$. *Then* I_α *is bounded from* $L^{p_0}_{s_0}$ *to* $L^{q_0}_{r_0}$.

In the unweighted case the following multilinear result is also known.

Fractional Integrals in Morrey Spaces: Known Results

Proposition C. (Tang) *Let* $0 < \alpha < mn$, $1 < q \le r < \infty$, $1 < p_i \le s_i < \infty$, $i = 1, \dots, m$ *be such that*

$$\frac{1}{s} - \frac{1}{r} = \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n},$$

where p *and* s *are defined by*

$$\frac{1}{p} := \sum_{i=1}^{m} \frac{1}{p_i}, \quad \frac{1}{s} := \sum_{i=1}^{m} \frac{1}{s_i}, \quad m \ge 2. \tag{0.5}$$

Then there exists a positive constant C *such that for all* $f_j \in L^{p_j}_{s_j}$, $j = 1, \dots, m$, *we have*

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^q_r} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{s_j}}.$$

Recall D. Adams Trace Theorem for for the Riesz Potentials I_α:

Theorem. *Let* $1 < p < q < \infty$ *and let* $0 < \alpha < n/p$. *Suppose that* μ *is a Borel measure on* $\mathbb{R}^n$. *Then the inequality*

$$\|I_\alpha(f)\|_{L^q(\mu)} \leq C\|f\|_{L^p}$$

holds if and only if

$$[\mu] := \sup_Q (\mu(Q))^{\frac{1}{q}} |Q|^{\frac{\alpha}{n} - \frac{1}{p}} < \infty. \tag{0.6}$$

Moreover, $\|I_\alpha\|_{L^p \mapsto L^q(\mu)} \approx [\mu]$.

We have analogous multilinear characterization is the following form:
Theorem B. [V. Kokilashvili, M. Mastylo and A. M. 2014] *Let* $1 < p_i < \infty$, $i = 1, \ldots, m$. *Assume that* $0 < \alpha < n/p$ *and* $p < q < \infty$. *Then the following assertions are equivalent:*
(i) For all f_i in L^{p_j} we have

$$\left\|\mathcal{I}_\alpha(\vec{f})\right\|_{L^q(V)} \leq C \prod_{i=1}^{m} \left\|f_i\right\|_{L^{p_j}};$$

(ii) the following condition holds:

$$[V]_{\alpha,p,q} := \sup_{Q\in\mathcal{Q}} \left(\int_Q V(x)dx\right)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n}-\frac{1}{p}} < \infty$$

Theorem. *Let* $1 < q \le r < \infty$, $1 < p_i \le s_i < \infty$, $i = 1, \ldots, m$, $1 < p < q$, $0 < \alpha < \frac{n}{s}$. *Let* $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n} - \frac{1}{\ell}$, *where* $\frac{1}{s} = \sum_{j=1}^{m} \frac{1}{s_j}$, $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$. *Then there exists a positive constant* C *depending only on* n, α, q, r, p_i, s_i, $i = 1, \ldots, m$, *such that for all* $f_j \in L_{s_j}^{p_j}$, $j = 1, \ldots, m$, *inequality*

$$\|g\,\mathcal{I}_\alpha(\vec{f}\,)\|_{L_r^q} \le C\|g\|_{L_\ell^q} \prod_{j=1}^{m} \|f_j\|_{L_{s_j}^{p_j}},$$

holds. Moreover, this estimate is sharp in the sense that we can not replace $\|g\|_{L_\ell^q}$ *by the smaller Morrey norm, i.e. we can not enlarge Morrey space here.*

Theorem. Let $1 < q \le r < \infty$, $1 < p_i \le s_i < \infty$, $i = 1, \dots, m$, $1 < p < q$, $0 < \alpha < \frac{n}{s}$. Let $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r}$, where $\frac{1}{s} = \sum_{j=1}^{m} \frac{1}{s_j}$, $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$. Suppose that V is a weight function on $\mathbb{R}^n$. Then the following statements are equivalent:

(i) there is a positive constant C such that for all measurable $\vec{f}$ we have

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^q_r(V)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{s_j}}. \tag{0.7}$$

(ii) Condition

$$[V]_{\alpha,p,q} := \sup_{Q \in \mathcal{Q}} \left(\int_Q V(x)dx \right)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n} - \frac{1}{p}} < \infty \tag{0.8}$$

is satisfied.

Moreover, under either assumption, we have the norm equivalence $\|\mathcal{I}_\alpha\| \approx [V]_{\alpha,p,q}$.

In the linear case, i.e., when $m = 1$, we have:

Corollary

Let $1 < q \leq r < \infty$, $1 < p \leq s < \infty$, $1 < p < q$ and $0 < \alpha < \frac{n}{s}$. Let $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n} - \frac{1}{\ell}$. Then there is a positive constant C depending only on n, α, q, r, p, s such that for all $f \in L^p_s$ and $g \in L^q_\ell$ we have

$$\left\| g\, I_\alpha(f) \right\|_{L^q_r} \leq C \left\| g \right\|_{L^q_\ell} \left\| f \right\|_{L^p_s}$$

Main Results

We also have a result for the bilinear fractional integral operator B_α.

Theorem

Let $1 < q \leq r$, $1 < p_i \leq s_i < \infty$, $i = 1, 2$. *Let* $1 < p < q < \infty$ *and* $0 < \alpha < \min\{\frac{1}{s_1}, \frac{1}{s_2}\}$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n} - \frac{1}{\ell}$, *where* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. *Then there is a positive constant* C *depending only on* n, α, q, r, r, p_1, p_2, s_1, s_2 *such that for all* $f_1, f_2, g \geq 0$ *we have*

$$\|g\, B_\alpha(f_1, f_2)\|_{L^q_r} \leq C\|g\|_{L^q_\ell}\|f_1\|_{L^{p_1}_{s_1}}\|f_2\|_{L^{p_2}_{s_2}}. \tag{0.9}$$

Main Results

Furthermore, we have the trace inequality for B_α which analogous to that of Adams (1971); see also (Eridani, V. Kokilashvili and A. M. for Morrey spaces) in the linear case.

Theorem

Let $1 < q \le r$, $1 < p_i \le s_i < \infty$, $i = 1, 2$, *and let* $1 < p < q < \infty$. *Let* $0 < \alpha < \min\{\frac{1}{s_1}, \frac{1}{s_2}\}$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r}$, *where* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. *Then there is a positive constant* C *depending on* n, α, q, r, p_1, p_2, s_1, s_2 *such that for all* $f_1, f_2 \ge 0$,

$$\|B_\alpha(f_1, f_2)\|_{L^q_r(d\mu)} \le C[\mu]\|f_1\|_{L^{p_1}_{s_1}}\|f_2\|_{L^{p_2}_{s_2}}, \tag{0.10}$$

holds, where $[\mu]$ *is defined in* (0.6).

Main Results

As a corollary we have the trace inequality for classical Lebesgue spaces.

Corollary

Let $1 < p_i < \infty$, $1 < p < q < \infty$ *and let* $0 < \alpha < \min\{\frac{1}{p_1}, \frac{1}{p_2}\}$. *Suppose that* μ *is a Borel measure on* $\mathbb{R}^n$. *Then there is a positive constant* C *such that for all* $f_1, f_2 \geq 0$,

$$\|B_\alpha(f_1, f_2)\|_{L^q(d\mu)} \leq C[\mu]\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}},$$

where $[\mu]$ *is defined in* (0.6).

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Trace inequality characterization in Morrey spaces:

Eridani, V. Kokilashvili and A. Meskhi, *Morrey spaces and fractional integral operators*, Expo. Math. **27** (2009), 227–239.

Trace inequality characterization for multilinear Riesz potentials:
V. Kokilashvili, M. Mastylo and A.Meskhi, On the boundedness of the multilinear fractional integral operators, *Nonlinear Analysis, Theory, Methods and Applications*, **94**(2014), 142-147.

Proof of the main result

First we recall the following statement:

Let $1 < p_i < \infty$, $i = 1, \ldots, m$. Assume that $\alpha < n/p$ and $p < q < \infty$. Then the following estimate holds:

$$\left\|\mathcal{I}_\alpha(\vec{f}\,)\right\|_{L^q(V)} \le C\,[V]_{\alpha,p,q} \prod_{j=1}^{m} \left\|f_j\right\|_{L^{p_j}}. \tag{0.11}$$

Proof of this Proposition is based on the next statement:

Lemma A. [V. Kokilashvili, M. Mastylo and A.M., 2014] *Let $1 < p_i < \infty$, $i = 1, \cdots, m$. Suppose that $0 < \alpha, \beta < n/p$ with the condition $\beta < \alpha$. There is a positive constant $C = C_{\alpha,\beta,p}$ such that for all non-negative $f_i \in L^{p_i}$, $i = 1, \ldots, m$, the pointwise estimate*

$$\mathcal{I}_\alpha(\vec{f})(x) \leq C\left[\left(\mathcal{M}_{\alpha-\beta}(\vec{f})(x)\right)^{\frac{\alpha-n/p}{\alpha-\beta-n/p}}\left(\prod_{i=1}^{m}\|f_i\|_{L^{p_i}}\right)^{\frac{\beta}{\beta-\alpha+n/p}}\right]$$

holds for all $x \in \mathbb{R}^n$.

Proposition D. ([Moen 2009]) *Let* $1 < p_i < \infty$, $i = 1, \dots, m$. *Assume that* $0 < \alpha < n/p$ *and* $p < q < \infty$. *Then the inequality*

$$\|\mathcal{M}_\alpha(\overrightarrow{f})\|_{L^q(V)} \le C \prod_{i=1}^{m} \left(\int\limits_{\mathbb{R}^n} \left|f_i(x)\right|^{p_i} dx \right)^{1/p_i}, \tag{0.12}$$

holds for the multilinear fractional maximal operator $\mathcal{M}_\alpha$ *if and only if* (0.8) *is satisfied. Moreover, if* C *is the best possible constant in* (0.12), *then* $C \approx [V]_{\alpha,p,q}$.

Proposition D is proved [Moen, 2009] in the two–weighted setting under the power-bump condition on weights but here we need that result only in a special case. Finally, for the purposes of this paper we need the following sharpening of Theorem B.

Proposition (Kokilashvili, Mastylo, Meskhi, 2014)

Let $1 < p_i < \infty$, $i = 1, \ldots, m$. *Assume that* $\alpha < n/p$ *and* $p < q < \infty$. *Then the following estimate holds:*

$$\left\|\mathcal{I}_\alpha(\vec{f})\right\|_{L^q(V)} \le C\,[V]_{\alpha,p,q} \prod_{j=1}^{m} \left\|f_j\right\|_{L^{p_j}}. \tag{0.13}$$

Proof. We adapt the arguments in [V. Kokilashvili, M. Mastylo and A.K, 2014]. Let β be as in Lemma A. We set

$$q_1 := q\frac{\alpha - \frac{n}{p}}{\alpha - \beta - \frac{n}{p}} = q\frac{\frac{\alpha}{n} - \frac{1}{p}}{\frac{\alpha-\beta}{n} - \frac{1}{p}}. \tag{0.14}$$

Then taking condition (0.8) and identity (0.14) into account we see that the following relations hold:

$$[V]^{q_1}_{\alpha-\beta,p,q_1} = \sup_{Q\in\mathcal{Q}} v(Q)|Q|^{((\alpha-\beta)/n-1/p)q_1} =$$

$$[V]^{q}_{\alpha,p,q} = \sup_{Q\in\mathcal{Q}} v(Q)|Q|^{(\alpha/n-1/p)q} < \infty.$$

Applying Lemma A and Proposition D we write

$$
\begin{aligned}
\|\mathcal{I}_\alpha(\vec{f})\|_{L^q(V)} &\le c_{\alpha,\beta,p}\left\|\mathcal{M}_{\alpha-\beta}(\vec{f})^{\frac{\alpha-n/p}{\alpha-\beta-n/p}}\right\|_{L^q(V)} \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{\beta}{\beta-\alpha+n/p}} \\
&= c_{\alpha,\beta,p}\|\mathcal{M}_{\alpha-\beta}(\vec{f})\|_{L^{q_1}(V)}^{q_1/q} \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{\beta}{\beta-\alpha+n/p}} \\
&\le c[V]_{\alpha-\beta,p,q_1}^{q_1/q} \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{q_1/q} \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{\beta}{\beta-\alpha+n/p}} \\
&= c[V]_{\alpha,p,q} \prod_{i=1}^m \|f_i\|_{L^{p_i}}.
\end{aligned}
$$

In the two equalities we used that $\frac{q_1}{q} = \frac{\alpha-\frac{n}{p}}{\alpha-\beta-\frac{n}{p}}$, which is a consequence of (0.14).

Proof of the main result

First observe that $p < q < \ell$ and $\frac{n}{\ell} < \alpha < \frac{n}{s} < \frac{n}{p}$. Without loss of generality we assume that $g \geq 0$, $f_j \geq 0$, $j = 1, \ldots, m$. For any ball $B := B(a, r)$, let $2B := B(a, 2r)$ be the ball with center a and radius $2r$. We write $f_j = f_j^0 + f_j^\infty$, where

$$f_j^0 = f_j \chi_{2B}, \;\; f_j^\infty = f_j \chi_{(2B)^c}, \;\; j = 1, \ldots, m.$$

Let $f_j \geq 0$, $j = 1, \ldots, m$. In view of this representation we write

$$\mathcal{I}_\alpha \vec{f}(x) \leq \mathcal{I}_\alpha(f_1^0, \ldots, f_m^0)(x) + \mathcal{I}_\alpha(f_1^\infty, \ldots, f_m^\infty)(x) + \sum_{j=1}^{m} \mathcal{I}_\alpha(f_1^{\beta_1}, \ldots, f_m^{\beta_m})(x),$$

where $\beta_1, \ldots, \beta_m \in \{0, \infty\}$ and the sum contains at least one $\beta_j = 0$ and $\beta_j = \infty$.

Consequently,

$$\left\|g\mathcal{I}_\alpha(\vec{f})\right\|_{L^q(B)} \le \left\|g\mathcal{I}_\alpha(f_1^0,\dots,f_m^0)\right\|_{L^q(B)} + \left\|gI_\alpha(f_1^\infty,\dots,f_m^\infty)\right\|_{L^q(B)} + \sum_{\beta_1,\dots,\beta_m} \left\|\mathcal{I}_\alpha(f_1^{\beta_1},\dots,f_m^{\beta_m})\right\|_{L^q(B)} := N_1 + N_2 + \sum.$$

Using Proposition 31 for $V = |g|^q$, we write

$$N_1 \le C\|g\|_{L^q_\ell} \prod_{j=1}^m \|\chi_{2B} f_j\|_{L^{p_j}} \le C\|g\|_{L^q_\ell} \prod_{j=1}^m \|\chi_{2B} f_j\|_{L^{p_j}_{s_j}} r^{n\sum_{j=1}^m \left(\frac{1}{p_j}-\frac{1}{s_j}\right)}$$
$$= C\|g\|_{L^q_\ell} \prod_{j=1}^m \|\chi_{2B} f_j\|_{M^{p_j}_{s_j}} r^{n\left(\frac{1}{p}-\frac{1}{s}\right)} = C\|g\|_{L^q_\ell} \prod_{j=1}^m \|\chi_{2B} f_j\|_{M^{p_j}_{s_j}} r^{n\left(\frac{1}{q}-\frac{1}{r}\right)}.$$

Proof

Let us estimate N_2. First observe that if $x \in B$ and $y_j \in (2B)^c$, then by simple geometric observations we find that $\frac{1}{2}|a - y_j| \leq |x - y_j| \leq \frac{3}{2}|a - y_j|$. Thus, we get

$$\mathcal{I}_\alpha(f_1^\infty, \dots, f_m^\infty)(x) \leq C \int_{2r}^{\infty} s^{\alpha - mn - 1} \left(\prod_{j=1}^{m} \int_{\{y_j : |x - y_j| < s\}} f_j^\infty(y_j) dy_j \right) ds$$

$$\leq C \int_{2r}^{\infty} s^{\alpha - mn - 1} \left(\prod_{j=1}^{m} \int_{\{y_j : |a - y_j| < 2s\}} f_j^\infty(y_j) dy_j \right) ds$$

$$\leq C \int_{2r}^{\infty} s^{\alpha - mn - 1} \left(\prod_{j=1}^{m} \|f_j\|_{L^{p_j}(B(a,2s))} \right) \prod_{j=1}^{m} (s^{n/p_j'}) ds$$

$$\leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{s_j}} \int_{2r}^{\infty} s^{\alpha - 1 - \sum_{j=1}^{m} \frac{n}{p_j} + n \sum_{j=1}^{m} \left(\frac{1}{p_j} - \frac{1}{s_j} \right)} ds$$

Proof

$$\left(\int_B \mathcal{I}_\alpha(f_1^\infty, \ldots, f_m^\infty)^q(x) g^q(x)dx\right)^{1/q} \leq Cr^{\alpha - \frac{n}{p} + n\left[\frac{1}{q} - \frac{1}{r}\right]}\left(\int_B g^q(x)dx\right)^{1/q} \prod_{j=}^{m}$$

$$\leq Cr^{n\left[\frac{1}{q} - \frac{1}{r}\right]} \|g\|_{L_\ell^q} \prod_{j=1}^{m} \|f_j\|_{L_{s_j}^{p_j}}.$$

In the last equality we used the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} - \frac{1}{\ell}$.

Proof

It remains to estimate $\sum$. For simplicity we take $m \geq 3$, $\beta_1 = \beta_2 = \infty$ and $\beta_3 = \cdots = \beta_m = 0$. Recall that $|x - y_j| \approx |a - y_j|$ for all $x \in B$ and $y_j \in (2B)^c$, $j = 1, 2$. Thus, without loss of generality, we have that one of the terms of Σ can be estimated as follows:

$$\begin{aligned}
&\mathcal{I}_\alpha(f_1^\infty, f_2^\infty, f_3^0, \ldots, f_m^0)(x) \\
&\quad = \int_{(2B)^c \times (2B)^c \times 2B \times \ldots \times 2B} \frac{f_1(y_1) f_2(y_2) f_3(y_3) \cdots f_m(y_m)}{(|x - y_1| + \cdots + |x - y_m|)^{mn-\alpha}} d\vec{y} \\
&\quad \leq C \left(\int_{(2B)^c \times (2B)^c} \frac{f_1(y_1) f_2(y_2) dy_1 dy_2}{(|a - y_1| + |a - y_2|)^{mn-\alpha}} \right) \left(\int_{(2B) \times \cdots \times (2B)} f_3(y_3) \cdots f_m(y_m
\end{aligned}$$

$$:= C I_1 \cdot I_2.$$

Proof

Now we estimate I_1 and I_2 separately. By Hölder's inequality and simple observations we obtain:

$$I_1 = C \int\limits_{(2B)^c \times (2B)^c} \left(\int\limits_{|a-y_1|+|a-y_2|}^{\infty} s^{-mn+\alpha-1} ds \right) f_1(y_1) f_2(y_2) dy_1 dy_2$$

$$\leq C \int\limits_{2r}^{\infty} \left(\int\limits_{\{y_1, y_2 : |a-y_1|+|a-y_2|<s\}} f_1(y_1) f_2(y_2) dy_1 dy_2 \right) s^{-mn+\alpha-1} ds$$

$$\leq C \int_{2r}^{\infty} \prod_{i=1}^{2} \left(\int_{B(a,s)} f_i^{p_i}(y_i) dy_i \right)^{1/p_i} s^{\frac{n}{p_1'} + \frac{n}{p_2'} - mn + \alpha - 1} ds$$

$$\leq C \int_{2r}^{\infty} \prod_{i=1}^{2} \left(\frac{1}{|B(a,s)|^{1-\frac{p_i}{s_i}}} \int\limits_{B(a,s)} f_i^{p_i}(y_i) dy_i \right)^{1/p_i} s^{\alpha - mn - 1 + n\left(\frac{1}{p_1'} + \frac{1}{p_2'}\right) + n\left[\frac{1}{p_1} + \frac{1}{p_1}\right.}$$

$$\leq C \prod^{2} \|f_i\|_{L^{p_i}_{s_i}} r^{\alpha - mn + 2n - n\left[\frac{1}{p_1} + \frac{1}{p_2}\right] + n\left[\frac{1}{p_1} - \frac{1}{s_1} + \frac{1}{p_2} - \frac{1}{s_2}\right]}.$$

Proof

In the latter estimate we used fact that

$$\alpha - mn + 2n - n\left(\frac{1}{p_1} + \frac{1}{p_2}\right) + n\left[\frac{1}{p_1} + \frac{1}{p_2} - \left(\frac{1}{s_1} + \frac{1}{s_2}\right)\right]$$
$$= \alpha - mn + 2n - n\left[\frac{1}{s_1} + \frac{1}{s_2}\right] < 0$$

which is a consequence of the condition $\alpha < \frac{n}{s}$. Further, by using Hölder's inequality again, we find that

$$I_2 \leq C \prod_{i=3}^{m} \|f_i\|_{L^{p_i}_{s_i}} r^{n\left(\sum_{k=3}^{m} \frac{1}{p'_k}\right) + n\left(\sum_{k=3}^{m}\left[\frac{1}{p_k} - \frac{1}{s_k}\right]\right)}$$
$$= C \prod_{i=3}^{m} \|f_i\|_{L^{p_i}_{s_i}} r^{n(m-2) - n\sum_{k=3}^{m} \frac{1}{p_k}} r^{n\left(\sum_{k=3}^{m}\left[\frac{1}{p_k} - \frac{1}{s_k}\right]\right)}.$$

Consequently, summarizing estimates for I_1 and I_2 we find that

Proof

$$\left(\int_B \mathcal{I}_\alpha(f_1^\infty, f_2^\infty f_3^0, \cdots, f_m^0)^q(x) g^q(x)dx\right)^{1/q} \leq Cr^{n\left[\frac{\alpha}{n}-\frac{1}{p}\right]} r^{n\left[\frac{1}{p}-\frac{1}{s}\right]+n\left[\frac{1}{q}-\frac{1}{\ell}\right]} \|g$$

$$= Cr^{n\left[\frac{1}{q}-\frac{1}{r}\right]} \|g\|_{L_\ell^q} \prod_{j=1}^m \|f_j\|_{L_{s_j}^{p_j}}.$$

In the last equality we again used the condition: $\frac{1}{p}-\frac{1}{q}=\frac{1}{s}-\frac{1}{r}=\frac{\alpha}{n}-\frac{1}{\ell}$.

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THANK YOU